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Hydrodynamic interaction of two unequal-sized spheres in a slightly rarefied gas: resistance and mobility functions

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The problem of the hydrodynamic interaction of two unequal-sized spheres in a slightly rarefied gas is treated following the singular perturbation scheme of Sone & Onishi (1978), valid at small, but finite, particle Knudsen numbers. In this method the solution to the linearized BGKW transport equation governing the gas molecular motion consists of two parts: one describing a Knudsen layer where the actual microscopic boundary conditions are applied and the other describing a Hilbert region where the Stokes equations of continuum hydrodynamics hold. The Knudsen-layer solution establishes the 'slip' boundary conditions for the Stokes equations. Here we clearly distinguish between particle 'slip' due to the type of boundary conditions and particle 'slip' due to lengthscale effects as measured by the Knudsen number. The present analysis has been carried out to first order in particle Knudsen number for the case of diffuse reflective molecular boundary conditions. General relationships between the first- and zero-order velocity fields, both of which are written in the form of Lamb's (1932) solution to the Stokes equation, are established. It is illustrated how these general relationships can be used to determine the force and torque acting on a single sphere translating and rotating in a slightly rarefied gas. Finally, we have treated the two-sphere problem in a slightly rarefied gas using the twin multipole expansion method of Jeffrey & Onishi (1984). Here again, general relationships are established between the solutions of the first-order fluid velocity field and the zero-order velocity field, the latter being shown to recover Jeffrey & Onishi's results for stick boundary conditions. These general relationships are subsequently used to determine the complete resistance and mobility matrices of the two-sphere system. The symmetric properties of the resistance and mobility matrices are demonstrated for slip boundary conditions, in agreement with the general proof of Landau & Lifshitz (1980) and Bedeaux, Albano & Mazur (1977).

1. Introduction

The problem of hydrodynamic interaction of two or more bodies in a gas is complicated compared with the interactions in a continuum fluid owing to the phenomenon of fluid 'slip' at solid boundaries. Because of this complication, few studies exist concerning the resistance and mobility functions for hydrodynamically interacting particles in a gas environment, even under low-Reynolds-number conditions for the gas. This is despite the fact that numerous practical problems exist involving interacting small particles in a gas, such as the transport behaviour of interacting aerosols, aerosol coagulation, and aerosol deposition onto surfaces (Marlow 1980).

Several theoretical studies exist that consider the hydrodynamic interactions of two spheres, and a sphere with a plane wall, under 'classic' mixed slip–stick boundary conditions (Davis 1972; Hocking 1973; Goren 1973; Felderhof 1977). Although there has been some suggestion that these analyses can provide slip corrections for hydrodynamic interactions in a gas environment (Dahneke 1974; Barnocky & Davis 1988) rigorous kinetic theory treatments demonstrate that the actual analysis and boundary conditions are more complex, even at small, but finite, particle Knudsen numbers (ratio of the gas mean free path to a characteristic particle length) where slip corrections are small (Cercignani 1975). We should also mention the kinetic theory study of Cukier, Kapral & Mehaffey (1981) on the hydrodynamic interaction of two spheres at large interparticle separations and the limit of vanishing Knudsen numbers. Those authors were able to recover classic full-slip results for specular reflective molecular boundary conditions. In the limit of zero Knudsen numbers, Sone & Akoi (1977) have also obtained classic full-slip results for a single sphere under specular reflective molecular boundary conditions and classic no-slip results for diffuse reflective molecular boundary conditions, following their kinetic theory method. Clearly these studies show that one must distinguish between particle 'slip' due to the type of boundary condition and particle 'slip' due to lengthscale effects as measured by the Knudsen number. The latter problem is the subject of the present work. The general problem is to solve the Boltzmann transport equation for the gas molecules subject to the actual microscopic boundary conditions, for example diffuse or specular boundary conditions, on solid surfaces (Cercignani 1975).

Recently, Onishi (1984) has used a previously developed singular perturbation method (Sone & Onishi 1978) to solve the boundary-valued Boltzmann transport equation for the axisymmetric problem of the so-called thermal-creep phenomenon of two interacting spheres in a slightly rarefied gas. In the present study, we follow a similar procedure to determine the hydrodynamic resistance functions of two unequal-sized spheres in a slightly rarefied gas.

The starting point in the analysis is the linearized BGKW (Bhatnager, Gross & Krook 1954; Welander 1954) form of the Boltzmann equation. A perturbation expansion of the linearized Boltzmann equation is considered in terms of the particle Knudsen number, Kn. In general, the problem is of the singular perturbation type involving a boundary or Knudsen layer on the particle surfaces and a continuum or Hilbert region outside the Knudsen layer. The Hilbert region has been shown to be governed by the usual Stokes equations of low-Reynolds-number hydrodynamics (Grad 1963). The essence of the procedure of Sone & Onishi is to first solve the inner or boundary-layer problem with the microscopic boundary conditions on particle surfaces. Asymptotic matching of the boundary-layer solution with the Hilbert region establishes the innermost boundary conditions necessary to uniquely solve the Stokes equations.

The determination of the hydrodynamic resistance and mobility functions for two interacting spheres thus still requires solution of the Stokes equations of continuum hydrodynamics once the proper boundary conditions have been established through the Knudsen-layer analysis. Here we follow the comprehensive analysis of Jeffrey & Onishi (1984), based on a generalization of the method of reflections technique (Happel & Brenner 1986) known as a twin multipole expansion method, in order to solve the Stokes equations subject to slip boundary conditions on the surfaces of two spheres. It is shown, however, that the application of Jeffrey & Onishi's method to the Knudsen-number expansion that describes particle slip requires some newly developed procedures. In fact, for the mobility functions it is shown that an innovative approach is necessary. In this study, we give expressions for the complete resistance and mobility matrix for the hydrodynamic interaction of two unequal-sized spheres at small, but finite, particle Knudsen numbers under diffuse reflective molecular boundary conditions that are shown to collapse to Jeffrey & Onishi's solution when the particle Knudsen number is zero.

The solutions given here are applicable up to the point of overlapping of the Knudsen layers surrounding each sphere, where the dimensionless thickness of the Knudsen layers is $O(Kn)$. The solution breaks down near the point of touching of the two spheres where a near-field analysis, analogous to lubrication theory solutions in liquids, would be required. For example, problems involving the coagulation and deposition of aerosols may require both the near-field and far-field solutions.

2. Hilbert region solution. General solutions of Stokes equations subject to slip boundary conditions

Following Sone and Onishi (Sone 1969; Sone & Onishi 1978; and the references therein) we consider an asymptotic solution to the linearized Bhatnager–Gross–Krook–Walender (BGKW) form of the Boltzmann transport equation at small but finite Knudsen numbers, $Kn = \lambda/L$, where λ is the mean free path of the gas molecules and L is some characteristic 'macroscopic' lengthscale. Diffuse reflection is assumed to describe the interaction of the gas with a bounding solid wall; i.e. (i) reflected molecules have a Maxwellian distribution characterized by the velocity and temperature of the wall, and (ii) the net mass flow of gas across the boundary is zero (no evaporation or condensation on the solid boundary). Here we only summarize the results necessary in the analysis of the isothermal two-sphere problem, relegating further details to the above-cited references.

Let f stand for any one of the fluid dynamic quantities such as velocity, pressure and temperature. It can be expressed by a sum of two parts, the Hilbert part f_{H}, which does not change appreciably over the mean free path of the gas, and the Knudsen-layer part f_{K}, which varies appreciably near the surface over a distance of the gas mean free path in the direction normal to the boundary, i.e. $f = f_{\mathrm{H}} + f_{\mathrm{K}}$. The Hilbert quantities can be obtained in an expansion in terms of the Knudsen number of the system as

$$f_{\mathrm{H}} = f^0_{\mathrm{H}} + K f^1_{\mathrm{H}} + K^2 f^2_{\mathrm{H}} + \dots, \tag{1}$$

where $K = \frac{1}{2}\pi^{\frac{1}{2}} Kn$. From a moment analysis of the BGKW linearized Boltzmann equation for the fluid molecules, it can be shown that the Hilbert parts of the fluid dynamic quantities satisfy Stokes equations to any order in the small-Knudsen-number expansion (Sone & Onishi 1978; also see Grad 1963). Explicitly these equations are

$$\nabla p^0_{\mathrm{H}} = 0, \tag{2}$$

$$\left.\begin{aligned} \nabla \cdot \boldsymbol{u}^i_{\mathrm{H}} &= 0, \qquad (3)\\ \nabla p^{i+1}_{\mathrm{H}} - \nabla^2 \boldsymbol{u}^i_{\mathrm{H}} &= 0, \qquad (4)\end{aligned}\right\} \quad (i = 0, 1, 2, \dots),$$

where $p_0(1+p)$ and $(2kT_0/m)\,\boldsymbol{u}$ are the pressure and the mean molecular velocity, respectively; p_0 and T_0 are the reference pressure and temperature, respectively; $\mathrm{L}^{-1}\nabla$ is the vector differential operator in physical space; k is Boltzmann's constant and m is the mass of a single molecule.

Sone & Onishi (1978) also show that solution of the BGKW linearized Boltzmann equation in the Knudsen layer near the surface, where diffuse molecular reflection is assumed to hold, leads to the boundary conditions appropriate for (2)–(4) on the interface between the gas and the solid wall as

$$\boldsymbol{u}_{\mathrm{H}}^{0} = \boldsymbol{u}_{\mathrm{W}}, \tag{5}$$

$$\boldsymbol{u}_{\mathrm{H}}^{1} \cdot \boldsymbol{n} = 0, \tag{6}$$

and

$$\boldsymbol{u}_{\mathrm{H}}^{1} \cdot \boldsymbol{t} = -\kappa_0 \boldsymbol{n} \cdot [\nabla \boldsymbol{u}_{\mathrm{H}}^{0} + (\nabla \boldsymbol{u}_{\mathrm{H}}^{0})^{\mathrm{T}}] \cdot \boldsymbol{t}, \tag{7}$$

where $\kappa_0 = -1.016191$, $\boldsymbol{u}_{\mathrm{W}}$ is the velocity of the solid boundary, $\boldsymbol{n}$ and $\boldsymbol{t}$ are the unit normal vector (towards the gas) and the unit tangential vector to the interface, respectively. Note that the zero-order boundary condition, (5), is the usual no-slip condition, whereas the first-order boundary conditions, given by (6) and (7), are the slip boundary conditions. Further note from (7) that the slip boundary conditions require the solution of the zero-order Hilbert solution at the gas–solid interface.

Since the Hilbert solutions satisfy the Stokes equations to any order of the particle Knudsen-number expansion, both the zero-order and first-order Hilbert solutions of the velocity field, $\boldsymbol{u}_{\mathrm{H}}^{0}$ and $\boldsymbol{u}_{\mathrm{H}}^{1}$, can be cast in the general form given by Lamb (1932). Outside a sphere with radius a, in spherical coordinates (r, θ, ϕ), they can be written as

$$\boldsymbol{u}_{\mathrm{H}}^{\delta} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \left\{ \nabla \times \left[r q_{mn}^{\delta} \left(\frac{a}{r}\right)^{n+1} Y_{mn}(\theta, \phi) \right] + a \nabla \left[v_{mn}^{\delta} \left(\frac{a}{r}\right)^{n+1} Y_{mn}(\theta, \phi) \right] \right.$$
$$\left. - \frac{n-2}{2n(2n-1)\,a} r^2 \nabla \left[p_{mn}^{\delta} \left(\frac{a}{r}\right)^{n+1} Y_{mn}(\theta, \phi) + \frac{n+1}{n(2n-1)\,a} r p_{mn}^{\delta} \left(\frac{a}{r}\right)^{n+1} Y_{mn}(\theta, \phi) \right] \right\}, \tag{8}$$

where $Y_{mn}(\theta, \phi)$ are surface spherical harmonics. Here and hereafter the superscript $\delta = 0$ or 1 represents the order in the particle Knudsen-number expansion.

When the boundary conditions are given in the form of a prescribed velocity field on the surface of the sphere, Happel & Brenner (1986) proposed a convenient way to determine the coefficients q_{mn}^{δ}, v_{mn}^{δ} and p_{mn}^{δ}. First, from the given boundary conditions $\boldsymbol{u}_{\mathrm{H}}^{\delta}|_{r=a} = \boldsymbol{U}^{\delta}(\theta, \phi)$, three scalar quantities X_{mn}^{δ}, Ψ_{mn}^{δ} and Z_{mn}^{δ} are constructed as

$$u_{\mathrm{H}r}^{\delta}|_{r=a} = U_r^{\delta} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} X_{mn}^{\delta} Y_{mn}(\theta, \phi), \tag{9}$$

$$r \frac{\partial u_{\mathrm{H}r}^{\delta}}{\partial r}\bigg|_{r=a} = -2U_r^{\delta} - \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}(\sin\theta\, U_\theta^{\delta}) - \frac{1}{\sin\theta} \frac{\partial U_\phi^{\delta}}{\partial\phi} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \Psi_{mn}^{\delta} Y_{mn}(\theta, \phi), \tag{10}$$

$$\boldsymbol{r} \cdot \nabla \times \boldsymbol{u}_{\mathrm{H}}^{\delta}|_{r=a} = \boldsymbol{r} \cdot \nabla \times \boldsymbol{U}^{\delta} = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}(\sin\theta\, U_\phi^{\delta}) - \frac{1}{\sin\theta} \frac{\partial U_\theta^{\delta}}{\partial\phi} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} Z_{mn}^{\delta} Y_{mn}(\theta, \phi), \tag{11}$$

where the subscripts r, θ, ϕ represent the components of a velocity in the directions of the coordinates r, θ, ϕ, respectively. Then, the three coefficients p_{mn}^{δ}, v_{mn}^{δ} and q_{mn}^{δ} in (8) are related to these three scalar quantities by

$$p_{mn}^{\delta} = \frac{2n-1}{n+1} [(n+2) X_{mn}^{\delta} + \Psi_{mn}^{\delta}], \tag{12}$$

$$v_{mn}^{\delta} = \frac{1}{2(n+1)} [n X_{mn}^{\delta} + \Psi_{mn}^{\delta}], \tag{13}$$

$$q_{mn}^{\delta} = \frac{1}{n(n+1)} Z_{mn}^{\delta}. \tag{14}$$

The zero-order Hilbert solution $\boldsymbol{u}_{\mathrm{H}}^0$ can be readily obtained by applying the 'stick' boundary conditions given by (5). As shown below, with a known zero-order Hilbert solution $\boldsymbol{u}_{\mathrm{H}}^0$ given in the form of (8), the boundary conditions satisfied by the first-order Hilbert solution $\boldsymbol{u}_{\mathrm{H}}^1$ can be determined from (6) and (7), which leads to direct relations between the coefficients p_{mn}^0, v_{mn}^0, q_{mn}^0 and p_{mn}^1, v_{mn}^1, q_{mn}^1.

First, (6) gives the value of $u_{\mathrm{H}r}^1$ on a sphere with radius a as

$$u_{\mathrm{H}r}^1|_{r=a} = 0. \tag{15}$$

Equations (7) and (8) give the values of $u_{\mathrm{H}\theta}^1$ and $u_{\mathrm{H}\phi}^1$ on the sphere in terms of $u_{\mathrm{H}r}^0$, $u_{\mathrm{H}\theta}^0$ and $u_{\mathrm{H}\phi}^0$ as

$$\begin{aligned} u_\phi^1|_{r=a} &= -\kappa_0 a \left[\frac{1}{r\sin\theta}\frac{\partial u_r^0}{\partial\phi} + \frac{\partial u_\phi^0}{\partial r} - \frac{u_\phi^0}{r}\right]\Bigg|_{r=a} \\ &= -\kappa_0 \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\left[-(n+2)\,q_{mn}^0 \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} Y_{mn}(\theta,\phi) - 2(n+2)\,v_{mn}^0 \frac{\partial}{\partial\theta} Y_{mn}(\theta,\phi)\right. \\ &\quad \left. + \frac{n^2-1}{n(2n-1)} p_{mn}^0 \frac{\partial}{\partial\theta} Y_{mn}(\theta,\phi)\right], \end{aligned} \tag{16}$$

and

$$\begin{aligned} u_\phi^1|_{r=a} &= -\kappa_0 a \left[\frac{1}{r\sin\theta}\frac{\partial u_r^0}{\partial\phi} + \frac{\partial u_\phi^0}{\partial r} - \frac{u_\phi^0}{r}\right]\Bigg|_{r=a} \\ &= -\kappa_0 \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\left[(n+2)\,q_{mn}^0 \frac{\partial}{\partial\theta} Y_{mn}(\theta,\phi) - 2(n+2)\,v_{mn}^0 \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} Y_{mn}(\theta,\phi)\right. \\ &\quad \left. + \frac{n^2-1}{n(2n-1)} p_{mn}^0 \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} Y_{mn}(\theta,\phi)\right]. \end{aligned} \tag{17}$$

Note that the particle radius a, that appears in the dimensional velocity fields of (16) and (17), is the result of selecting the characteristic lengthscale $L = a$ in the dimensionless boundary condition (7). Then from (9)–(11) the three scalar quantities X_{mn}^1, Ψ_{mn}^1 and Z_{mn}^1 are found to be

$$X_{mn}^1 = 0, \tag{18}$$

$$\Psi_{mn}^1 = \kappa_0\left[2n(n+1)(n+2)\,v_{mn}^0 - \frac{(n+1)(n^2-1)}{2n-1} p_{mn}^0\right], \tag{19}$$

$$Z_{mn}^1 = \kappa_0\, n(n+1)(n+2)\, q_{mn}^0. \tag{20}$$

Substituting (18)–(20) into (12)–(14) further leads to the relations between p_{mn}^0, v_{mn}^0, q_{mn}^0 and p_{mn}^1, v_{mn}^1, q_{mn}^1 as

$$p_{mn}^1 = \kappa_0[2n(2n-1)(n+2)\,v_{mn}^0 - (n^2-1)\,p_{mn}^0], \tag{21}$$

$$v_{mn}^1 = \kappa_0\left[n(n+2)\,v_{mn}^0 - \frac{(n^2-1)}{2(2n-1)} p_{mn}^0\right], \tag{22}$$

$$q_{mn}^1 = \kappa_0 (n+2)\, q_{mn}^0. \tag{23}$$

Once the boundary-valued problem for flow external to a sphere has been solved, the frictional force $\boldsymbol{F}$ and the torque $\boldsymbol{T}_0$ (about the sphere centre) experienced by the sphere can be calculated. As shown by Sone & Tanaka (1980), the Knudsen solution does not contribute to the total force acting on the sphere. Following their line of proof, it can be shown that the Knudsen solution does not contribute to the total torque acting on the sphere either. Therefore, the Hilbert solution alone is sufficient for the purpose of calculating $\boldsymbol{F}$ and $\boldsymbol{T}_0$. According to the formulae given by Happel & Brenner (1986), taking into account the contributions from both zero-order and first-order Hilbert solutions, we have

$$\boldsymbol{F} = \boldsymbol{F}^0_{\mathrm{H}} + K\boldsymbol{F}^1_{\mathrm{H}} = -4\pi\mu p^0_{m1}\, a\boldsymbol{\nabla}[rY_{m1}(\theta,\phi)] - 4\pi K\mu p^1_{m1}\, a\boldsymbol{\nabla}[rY_{m1}(\theta,\phi)], \tag{24}$$

and

$$\boldsymbol{T} = \boldsymbol{T}^0_{\mathrm{H}} + K\boldsymbol{T}^1_{\mathrm{H}} = -8\pi\mu q^0_{m1}\, a^2\boldsymbol{\nabla}[rY_{m1}(\theta,\phi)] - 8\pi K\mu q^1_{m1}\, a^2\boldsymbol{\nabla}[rY_{m1}(\theta,\phi)]. \tag{25}$$

We now apply the general theory described above to two special cases in which a solid sphere is moving translationally or rotationally in an unbounded fluid otherwise at rest. Then, we apply the above results to the two-sphere problem.

3. Translation and rotation of a single sphere in a slightly rarefied gas

Case 1. A sphere with radius a is moving in the positive z-direction with a uniform velocity U.

From the boundary conditions for the zero-order Hilbert solution given by (5), we have

$$X^0_{mn} = U\delta_{m0}\,\delta_{n1}, \tag{26}$$

$$\Psi^0_{mn} = 0, \tag{27}$$

$$Z^0_{mn} = 0. \tag{28}$$

Substituting (26)–(28) into (12)–(14) yields

$$p^0_{mn} = \tfrac{3}{2}U\delta_{m0}\,\delta_{n1}, \tag{29}$$

$$v^0_{mn} = \tfrac{1}{4}U\delta_{m0}\,\delta_{n1}, \tag{30}$$

$$q^0_{mn} = 0. \tag{31}$$

Therefore, from (8), the zero-order Hilbert solution of the velocity field is

$$\boldsymbol{u}^0_{\mathrm{H}} = \tfrac{1}{4}a^3U\boldsymbol{\nabla}\left(\frac{\cos\theta}{r^2}\right) + \tfrac{3}{4}aU\boldsymbol{\nabla}\cos\theta + \frac{3}{2}\frac{a}{r^2}U\cos\theta\boldsymbol{r}. \tag{32}$$

Note that the zero-order solution is the well-known Stokes velocity field for no-slip boundary conditions.

To obtain the Knudsen-layer correction we substitute (29)–(31) into (21)–(23) to give

$$p^1_{mn} = \tfrac{3}{2}\kappa_0\, U\delta_{m0}\,\delta_{n1}, \tag{33}$$

$$v^1_{mn} = \tfrac{3}{4}\kappa_0\, U\delta_{m0}\,\delta_{n1}, \tag{34}$$

$$q^1_{mn} = 0. \tag{35}$$

Substitution of (33)–(35) into (8) gives the first-order slip correction for the Stokes velocity field as

$$\boldsymbol{u}^1_{\mathrm{H}} = \kappa_0\left[\tfrac{3}{4}a^3U\boldsymbol{\nabla}\left(\frac{\cos\theta}{r^2}\right) + \tfrac{3}{4}aU\boldsymbol{\nabla}\cos\theta + \frac{3}{2}\frac{a}{r^2}U\cos\theta\boldsymbol{r}\right]. \tag{36}$$

Case 2. The sphere is rotating in a counterclockwise direction about the z-axis with a fixed angular velocity $\boldsymbol{\Omega}$. Thus, with the boundary condition $\boldsymbol{u}^0_{\mathrm{H}} = \boldsymbol{\Omega}\times\boldsymbol{r}_0$ on the surface of the sphere, where $\boldsymbol{r}_0$ is the position vector of a point on the sphere surface relative to the centre of the sphere of radius a, we have

$$p^0_{mn} = 0, \tag{37}$$

$$v^0_{mn} = 0, \tag{38}$$

$$q^0_{mn} = a\Omega\delta_{m0}\,\delta_{n1}, \tag{39}$$

which gives the zero-order Hilbert solution of the velocity field as

$$\boldsymbol{u}^0_{\mathrm{H}} = a^3\Omega\nabla\times\left[\frac{\boldsymbol{r}}{r^2}\cos\theta\right]. \tag{40}$$

Proceeding as in Case 1, from (21)–(23) we obtain

$$p^1_{mn} = 0, \tag{41}$$

$$v^1_{mn} = 0, \tag{42}$$

$$q^1_{mn} = 3\kappa_0\,a\Omega\delta_{m0}\,\delta_{n1}, \tag{43}$$

which leads to the first-order Hilbert solution of the velocity field as

$$\boldsymbol{u}^1_{\mathrm{H}} = 3\kappa_0\,a^3\Omega\nabla\times\left[\frac{\boldsymbol{r}}{r^2}\cos\theta\right]. \tag{44}$$

From the force and torque equations (24) and (25) and the results in (29), (33), (39) and (43) we can also compute the force acting on the sphere from the first case and the torque from the second case as

$$\boldsymbol{F} = -6\pi\mu a[1+K\kappa_0]\,\boldsymbol{U}, \tag{45}$$

and

$$\boldsymbol{T}_0 = -8\pi\mu a^3[1+3K\kappa_0]\,\boldsymbol{\Omega}. \tag{46}$$

Both these expressions collapse to continuum hydrodynamic results for $K = 0$ (cf. Happel & Brenner 1986). Since $\kappa_0 < 0$, both the force and torque acting on the sphere are reduced owing to small but finite K. Equation (45) for the force acting on the particle has been previously given by Sone & Aoki (1977) and has been shown to give good agreement with experimental values at small Knudsen numbers and where the diffuse microscopic boundary condition is expected to hold (Brock 1980). The result for the torque acting on the particle given by (46) is believed to be new.

4. Hydrodynamic interaction of two unequal-sized spheres

In the following, the hydrodynamic interaction between two unequal rigid spheres which translate or rotate in an arbitrary manner in an unbounded fluid will be studied. The hydrodynamic interaction here refers to the mutual influence of the two bodies via the fluid. In other words, the relations between the force and torque that the fluid exerts on each sphere and the velocity and angular velocity of each sphere will be studied. The distinction between our study and the previous works on the same subject lies in that the dynamics in the Knudsen-layer region is investigated and its correction to the continuum hydrodynamic solutions is taken into account.

A precise definition of the interactions to be studied is as follows. Two rigid spheres, labelled sphere 1 and sphere 2, are immersed in an unbounded fluid characterized by a uniform velocity field $\boldsymbol{U}(\boldsymbol{r}) = \boldsymbol{U}_0 + \boldsymbol{\Omega}_0\times\boldsymbol{r}$ in the absence of the two spheres, i.e. a superposition of a uniform field and rigid body rotation. Sphere α

($\alpha = 1$ or 2) has radius a_α and its centre is at $\boldsymbol{r}_\alpha$; it has an angular velocity $\boldsymbol{\Omega}_\alpha$ and its centre has translational velocity $\boldsymbol{U}_\alpha$. The force the fluid exerts on sphere α is $\boldsymbol{F}_\alpha$ and the torque on sphere α relative to its centre is $\boldsymbol{T}_\alpha$.

4.1. *The resistance matrix*

If the specified quantities are the translational and angular velocities of two spheres and the velocity of the ambient flow, one can evaluate the forces and torques exerted by the fluid on the spheres. On account of the linearity of the Stokes equations, the above pairs are linearly related according to

$$\begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \end{bmatrix} = \mu \begin{bmatrix} A_{11} & A_{12} & \tilde{B}_{11} & \tilde{B}_{12} \\ A_{21} & A_{22} & \tilde{B}_{21} & \tilde{B}_{22} \\ B_{11} & B_{12} & C_{11} & C_{12} \\ B_{21} & B_{22} & C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_1 - \boldsymbol{U}_0 \\ \boldsymbol{U}_2 - \boldsymbol{U}_0 \\ \boldsymbol{\Omega}_1 - \boldsymbol{\Omega}_0 \\ \boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_0 \end{bmatrix}. \tag{47}$$

The square matrix is the resistance matrix, whose elements depend on the geometry of the system only and obey a number of symmetry conditions.

Based on Onsager's reciprocal relations via the thermodynamics of irreversible processes, Landau & Lifshitz (1980) have proven the symmetry of the resistance matrix. Since the usual equations of fluid dynamics are never invoked in their proof, the conclusion should hold for more general boundary conditions than no-slip boundary conditions. Also Bedeaux, Albano & Mazur (1977) provide a general proof of the symmetry of the resistance matrix with arbitrary slip boundary conditions based on hydrodynamics.

In addition to its being symmetrical, further symmetric properties are brought to the resistance matrix because of the fact that the system of two spheres is geometrically a body with rotational symmetry about the line of centres. Brenner (1963, 1964) shows that for such a system, each tensor in the resistance matrix can be reduced to an expression containing at most two scalar functions. Choosing the vector $\boldsymbol{l} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ along the positive z-direction in Cartesian coordinates and denoting the unit vectors along positive x-, y- and z-directions by $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$, respectively, the structure of these tensors determined by Brenner can be illustrated by dyadics as

$$A_{\alpha\beta} = X^A_{\alpha\beta}\hat{\boldsymbol{k}}\hat{\boldsymbol{k}} + Y^A_{\alpha\beta}(\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}\hat{\boldsymbol{\jmath}}), \tag{48}$$

$$B_{\alpha\beta} = Y^B_{\alpha\beta}(\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\jmath}}\hat{\boldsymbol{\imath}}), \tag{49}$$

$$C_{\alpha\beta} = X^C_{\alpha\beta}\hat{\boldsymbol{k}}\hat{\boldsymbol{k}} + Y^C_{\alpha\beta}(\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}\hat{\boldsymbol{\jmath}}). \tag{50}$$

Finally, by observation of the two-sphere geometry, the sphere labels 1 and 2 on the tensors' elements are interchangeable. Explicitly, if the sphere label changes from α to $3-\alpha$ any element in these tensors $E_{\alpha\beta}$ which is a function of a_1, a_2 and $\boldsymbol{l}$ only, obeys

$$E_{\alpha\beta}(a_1, a_2, \boldsymbol{l}) = E_{(3-\alpha)(3-\beta)}(a_2, a_1, -\boldsymbol{l}). \tag{51}$$

In summary, after applying all the above-mentioned symmetry conditions, the task of a complete determination of the resistance matrix reduces to an evaluation of ten scalar functions, which can be conveniently chosen as: X^A_{11}, X^A_{21}, Y^A_{11}, Y^A_{21}, Y^B_{11}, Y^B_{21}, X^C_{11}, X^C_{21}, Y^C_{11} and Y^C_{21}.

The method of reflections is used to determine the resistance and mobility matrices of the two-sphere problem. This method has the advantage of providing results with transparency and in a convenient form for further use in applications. The results gained from this method are reported to be numerically accurate for all but the

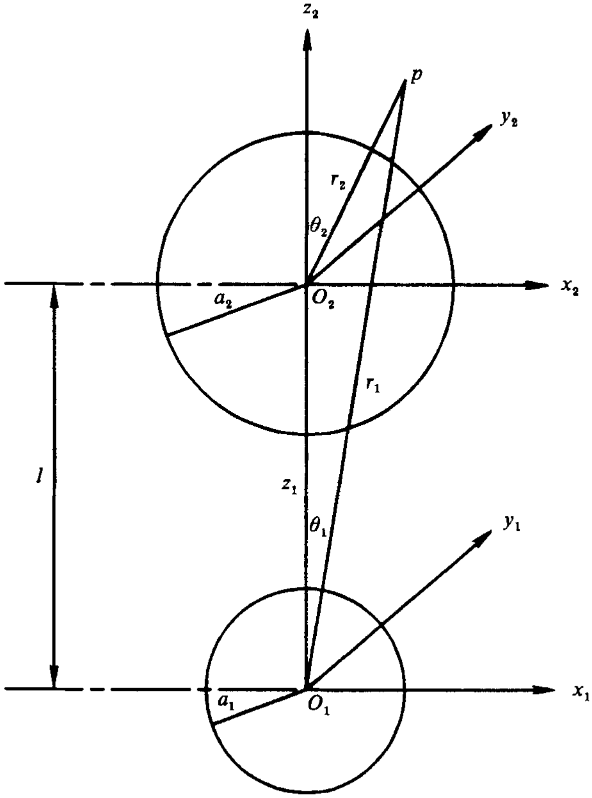

FIGURE 1. Coordinate system for two spheres.

nearest distances, and therefore should suffice for most applications (Jeffrey & Onishi 1984; Felderhof 1977). Anyway, the small-Knudsen-number expansion formulae assumed here will not hold at close interparticle separation distances (overlapping Knudsen layers on the two particles).

In order to apply the method of reflections, the two-sphere system is described by two sets of spherical coordinates $(r_\alpha, \theta_\alpha, \phi)$ $(\alpha = 1,2)$ shown in figure 1. The transformation rules between these two coordinate systems are (Hobson 1931)

$$\left(\frac{a_\alpha}{r_\alpha}\right)^{n+1} Y_{mn}(\theta_\alpha, \phi) = (-1)^{(n+m)(\alpha-1)} \left(\frac{a_\alpha}{l}\right)^{n+1} \sum_{s=m}^{\infty} (-1)^{(s+m)(2-\alpha)} \begin{bmatrix} n+s \\ s+m \end{bmatrix} \left(\frac{r_{3-\alpha}}{l}\right)^{s} Y_{ms}(\theta_{3-\alpha}, \phi), \tag{52}$$

$$\boldsymbol{r}_\alpha = [r_{3-\alpha} - (-1)^\alpha l \cos\theta_{3-\alpha}]\,\hat{\boldsymbol{r}}_{3-\alpha} + (-1)^\alpha l \sin\theta_{3-\alpha}\,\hat{\boldsymbol{\theta}}_{3-\alpha}, \tag{53}$$

$$r_\alpha^2 = r_{3-\alpha}^2 + l^2 - (-1)^\alpha 2 r_{3-\alpha} l \cos\theta_{3-\alpha}, \tag{54}$$

where $\hat{\boldsymbol{r}}_\alpha$ and $\hat{\boldsymbol{\theta}}_\alpha$ are the unit vectors in the coordinate directions.

The Hilbert velocity field outside the two spheres is $\boldsymbol{u}_{\mathrm{H}} = \boldsymbol{u}_{\mathrm{H}}^0 + K\boldsymbol{u}_{\mathrm{H}}^1$. While applying the method of reflections (Happel & Brenner 1986), $\boldsymbol{u}_{\mathrm{H}}^\delta$ consists of two parts $\boldsymbol{u}_{\mathrm{H}}^\delta = \boldsymbol{u}_{\mathrm{H}}^\delta(1) + \boldsymbol{u}_{\mathrm{H}}^\delta(2)$, where $\boldsymbol{u}_{\mathrm{H}}^\delta(\alpha)$ is given by (8) in the coordinates $(r_\alpha, \theta_\alpha, \phi)$ and the coefficients there are labelled by α as $p_{mn}^\delta(\alpha)$, $v_{mn}^\delta(\alpha)$ and $q_{mn}^\delta(\alpha)$. For clarity, the three scalar quantities introduced in (9)–(11) are also labelled by α as $X_{mn}^\delta(\alpha)$, $\Psi_{mn}^\delta(\alpha)$ and $Z_{mn}^\delta(\alpha)$, corresponding to the boundary conditions on the surface of sphere α.

In an exact manner, using the general expression for $\boldsymbol{u}_{\mathrm{H}}^{\delta}(1)$ and $\boldsymbol{u}_{\mathrm{H}}^{\delta}(2)$ given by (8) in conjunction with the transformation rules in (52)–(54), the coefficients $p_{mn}^{\delta}(\alpha)$, $v_{mn}^{\delta}(\alpha)$, $q_{mn}^{\delta}(\alpha)$ and the quantities $X_{mn}^{\delta}(\alpha)$, $\Psi_{mn}^{\delta}(\alpha)$, $Z_{mn}^{\delta}(\alpha)$ are linked by the following equations:

$$(n+1)(2n+1)\,v_{mn}^{\delta}(\alpha)-\frac{n+1}{2}p_{mn}^{\delta}(\alpha)$$
$$+\frac{n}{2n+3}(-1)^{(n+m)(\alpha-1)}\sum_{s=m}^{\infty}(-1)^{(s+m)(2-\alpha)}\begin{bmatrix}n+s\\n+m\end{bmatrix}p_{ms}^{\delta}(3-\alpha)\,\xi_{\alpha}^{n+1}\xi_{3-\alpha}^{s}$$
$$=\Psi_{mn}^{\delta}(\alpha)-(n-1)\,X_{mn}^{\delta}(\alpha), \tag{55}$$

$$\frac{n+1}{2n-1}p_{mn}^{\delta}(\alpha)+(-1)^{(n+m)(\alpha-1)}\sum_{s=m}^{\infty}(-1)^{(s+m)(2-\alpha)}\begin{bmatrix}n+s\\n+m\end{bmatrix}$$
$$\times\Big[(-1)^{3-\alpha}\mathrm{i}m(2n+1)\,q_{ms}^{\delta}(3-\alpha)\,\xi_{\alpha}^{n-1}\xi_{3-\alpha}^{s+1}$$
$$+n(2n+1)\,v_{ms}^{\delta}(3-\alpha)\,\xi_{\alpha}^{n-1}\xi_{3-\alpha}^{s+2}$$
$$+\frac{2n+1}{2n-1}\frac{ns(n+s-2ns-2)-m^{2}(2ns-4s-4n+2)}{2s(2s-1)(n+s)}p_{ms}^{\delta}(3-\alpha)\,\xi_{\alpha}^{n-1}\xi_{3-\alpha}^{s}$$
$$+\tfrac{1}{2}np_{ms}^{\delta}(3-\alpha)\,\xi_{\alpha}^{n+1}\xi_{3-\alpha}^{s}\Big]=\Psi_{mn}^{\delta}(\alpha)+(n+2)\,X_{mn}^{\delta}(\alpha), \tag{56}$$

$$n(n+1)\,q_{mn}^{\delta}(\alpha)+(-1)^{(n+m)(\alpha-1)}\sum_{s=m}^{\infty}(-1)^{(s+m)(2-\alpha)}\begin{bmatrix}n+s\\n+m\end{bmatrix}\Big[-nsq_{ms}^{\delta}(3-\alpha)\,\xi_{\alpha}^{n}\,\xi_{3-\alpha}^{s+1}$$
$$+\frac{(-1)^{3-\alpha}\mathrm{i}m}{s}p_{ms}^{\delta}(3-\alpha)\,\xi_{\alpha}^{n}\,\xi_{3-\alpha}^{s}\Big]=Z_{mn}^{\delta}(\alpha). \tag{57}$$

Here the notation $\xi_{\alpha}=(a_{\alpha}/l)$ has been introduced. The quantities $X_{mn}^{\delta}(\alpha)$, $\Psi_{mn}^{\delta}(\alpha)$ and $Z_{mn}^{\delta}(\alpha)$ in (55)–(57) are given by the boundary conditions satisfied by $\boldsymbol{u}_{\mathrm{H}}^{\delta}$ on sphere α through (9)–(11). Equations (55)–(57) will be useful in the calculation of both resistance and mobility functions under slip boundary conditions. Although (55)–(57) cannot be solved exactly, they can be approximately solved by appealing to the method of reflections as described below.

Now, slightly modifying Jeffrey & Onishi (1984), we introduce the following double power series expansions of the coefficients $p_{mn}^{\delta}(\alpha)$, $v_{mn}^{\delta}(\alpha)$ and $q_{mn}^{\delta}(\alpha)$:

$$p_{mn}^{\delta}(\alpha)=\tfrac{3}{2}\kappa_{0}^{\delta}(-1)^{3-\alpha}(-1)^{(n+m)(\alpha-1)}U\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}P_{npq}^{\delta}(\alpha)\,\xi_{\alpha}^{p}\,\xi_{3-\alpha}^{q}, \tag{58}$$

$$v_{mn}^{\delta}(\alpha)=\tfrac{3}{4}\kappa_{0}^{\delta}(-1)^{3-\alpha}(-1)^{(n+m)(\alpha-1)}U\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{2n+1}V_{npq}^{\delta}(\alpha)\,\xi_{\alpha}^{p}\,\xi_{3-\alpha}^{q}, \tag{59}$$

$$q_{mn}^{\delta}(\alpha)=\mathrm{i}\kappa_{0}^{\delta}(-1)^{3-\alpha}(-1)^{(n+m)(\alpha-1)}U\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}Q_{npq}^{\delta}(\alpha)\,\xi_{\alpha}^{p}\,\xi_{3-\alpha}^{q}. \tag{60}$$

It can be shown that the expansions (58)–(60) will result in recurrence formulas of $P_{npq}^{\delta}(\alpha)$, $V_{npq}^{\delta}(\alpha)$ and $Q_{npq}^{\delta}(\alpha)$ valid for both zero-order and first-order quantities in the K-expansion. Setting the right-hand sides of (55)–(57) to zero as dictated in the method of reflections procedure and substituting the expansions (58)–(60) into (55)–(57), then equating the coefficients of the same power of ξ_{α} and $\xi_{3-\alpha}$, the following recurrence relations for the pure numbers $P^{\delta}_{npq}(\alpha)$, $V^{\delta}_{npq}(\alpha)$ and $Q^{\delta}_{npq}(\alpha)$ can be derived:

$$P^{\delta}_{npq}(\alpha) = \sum_{s=1}^{q} \begin{bmatrix} n+s \\ n+m \end{bmatrix} \left[\frac{2n+1}{2(n+1)} \right. \times \frac{ns(n+s-2ns-2)-m^2(2ns-4s-4n+2)}{s(n+s)(2s-1)} P^{\delta}_{s(q-s)(p-n+1)}(3-\alpha)$$
$$+\frac{n(2n-1)}{2(n+1)} P^{\delta}_{s(q-s)(p-n-1)}(3-\alpha) + \frac{n(4n^2-1)}{2(n+1)(2s+1)} V^{\delta}_{s(q-s-2)(p-n+1)}(3-\alpha)$$
$$\left. -\frac{2(4n^2-1)}{3(n+1)} Q^{\delta}_{s(q-s-1)(p-n+1)}(3-\alpha) \right], \tag{61}$$

$$V^{\delta}_{npq}(\alpha) = P^{\delta}_{npq}(\alpha) + \frac{2n}{(n+1)(2n+3)} \sum_{s=1}^{q} \begin{bmatrix} n+s \\ n+m \end{bmatrix} P^{\delta}_{s(q-s)(p-n-1)}(3-\alpha), \tag{62}$$

$$Q^{\delta}_{npq}(\alpha) = \sum_{s=1}^{q} \begin{bmatrix} n+s \\ n+m \end{bmatrix} \left[\frac{s}{n+1} Q^{\delta}_{s(q-s-1)(p-n)}(3-\alpha) - \frac{3m}{2ns(n+1)} P^{\delta}_{s(q-s)(p-n)}(3-\alpha) \right]. \tag{63}$$

As long as the initial values of these pure numbers $P^{\delta}_{npq}(\alpha)$, $V^{\delta}_{npq}(\alpha)$ and $Q^{\delta}_{npq}(\alpha)$ are given, the remaining values can then be readily evaluated on a computer through the simultaneous solution of (61)–(63). To obtain the initial values of the pure numbers $P^{\delta}_{npq}(\alpha)$, $V^{\delta}_{npq}(\alpha)$ and $Q^{\delta}_{npq}(\alpha)$, we must set all the coefficients $p^{\delta}_{ms}(3-\alpha)$, $v^{\delta}_{ms}(3-\alpha)$ and $q^{\delta}_{ms}(3-\alpha)$ in (55)–(57) to zero and substitute the expansions (58)–(60) as well as the expressions for $X^{\delta}_{mn}(\alpha)$, $\Psi^{\delta}_{mn}(\alpha)$ and $Z^{\delta}_{mn}(\alpha)$ into (55)–(57) as dictated by the initial boundary conditions in the method of reflections procedure. Then equating the coefficients of the same power of ξ_{α} and $\xi_{3-\alpha}$ leads to the initial values for $P^{\delta}_{npq}(\alpha)$, $V^{\delta}_{npq}(\alpha)$ and $Q^{\delta}_{npq}(\alpha)$.

A special discussion is needed, however, concerning the slip boundary conditions satisfied by the first-order Hilbert solution in the case involving a two-sphere system. Generally, these boundary conditions are determined by the zero-order Hilbert solutions. Note that the zero-order Hilbert solution $\boldsymbol{u}^0_{\mathrm{H}}$ consists of a sum of two parts, i.e. $\boldsymbol{u}^0_{\mathrm{H}} = \boldsymbol{u}^0_{\mathrm{H}}(1) + \boldsymbol{u}^0_{\mathrm{H}}(2)$. The contributions to the boundary conditions satisfied by $\boldsymbol{u}^1_{\mathrm{H}}$ on sphere α from the part $\boldsymbol{u}^0_{\mathrm{H}}(\alpha)$ can be readily obtained from (18)–(20). The part $\boldsymbol{u}^0_{\mathrm{H}}(3-\alpha)$, however, must be converted into the $(r_{\alpha}, \theta_{\alpha}, \phi)$ system first, then, by applying a similar but more lengthy procedure as the one used to derive (18)–(20), its contributions to the boundary conditions for $\boldsymbol{u}^1_{\mathrm{H}}$ on sphere α can be derived. Adding the contributions from these two parts together completes the derivation of the boundary conditions for $\boldsymbol{u}^1_{\mathrm{H}}$ on sphere α. As a result, the three scalar quantities $X^1_{mn}(\alpha)$, $\Psi^1_{mn}(\alpha)$ and $Z^1_{mn}(\alpha)$ are given in terms of the coefficients $p^0_{mn}(\alpha)$, $v^0_{mn}(\alpha)$ and $q^0_{mn}(\alpha)$ by

$$X^1_{mn}(\alpha) = 0, \tag{64}$$

$$\Psi^1_{mn}(\alpha) = \kappa_0 \left(\frac{a_1}{a_2}\right)^{\alpha-1} \left\{ -\frac{(n+1)(n^2-1)}{2n-1} p^0_{mn}(\alpha) + 2n(n+1)(n+2) v^0_{mn}(\alpha) \right.$$
$$-(-1)^{(n+m)(\alpha-1)} \sum_{s=m}^{\infty} (-1)^{(s+m)(2-\alpha)} \begin{bmatrix} n+s \\ n+m \end{bmatrix} \left[q^0_{ms}(3-\alpha)(-1)^{\alpha} 2im(n^2-1)\, \xi_{\alpha}^{n-1} \xi_{3-\alpha}^{s+1} \right.$$

$$-v^0_{ms}(3-\alpha)\,2n(n^2-1)(n+1)\,\xi_\alpha^{n-1}\xi_{3-\alpha}^{s+2}-p^0_{ms}(3-\alpha)\frac{n^2(n+2)}{(2n+3)}\xi_\alpha^{n+1}\xi_{3-\alpha}^{s}$$
$$-p^0_{ms}(3-\alpha)\frac{(n^2-1)[ns(n+s-2ns-2)-2m^2(ns-2n-2s+1)]}{s(2s-1)(n+s)(2n-1)}\xi_\alpha^{n+1}\xi_{3-\alpha}^{s}\Bigg]\Bigg\}, \quad (65)$$

$$Z^1_{mn}(\alpha)=\kappa_0\left(\frac{a_1}{a_2}\right)^{\alpha-1}\Bigg\{n(n+1)(n+2)^2 q^0_{mn}(\alpha)+(-1)^{(n+m)(\alpha-1)}\sum_{s=m}^{\infty}(-1)^{(s+m)(2-\alpha)}\begin{bmatrix}n+s\\n+m\end{bmatrix}$$
$$\left[q^0_{mn}(3-\alpha)\,ns(n-1)(n+2)\,\xi_\alpha^n\xi_{3-\alpha}^{s+1}-p^0_{ms}(3-\alpha)(-1)^\alpha\frac{im(n-1)(n+2)}{s}\xi_\alpha^n\xi_{3-\alpha}^s\right]\Bigg\}. \quad (66)$$

Note that in the small-Knudsen-number analysis of the two-sphere problem, the characteristic length is taken to be the radius of the smaller sphere ($a_1 < a_2$). Thus, the inclusion of the factors $(a_1/a_2)^{\alpha-1}$ in (65) and (66) is the result of the definition of the Knudsen number as $Kn = \lambda/a_1$.

Following the zero-order procedures, (64)–(66) can be used to give the initial values of the pure numbers $P^1_{npq}(\alpha)$, $V^1_{npq}(\alpha)$ and $Q^1_{npq}(\alpha)$, which are needed to carry out the iterative computations in (61)–(63), in terms of the pure numbers $P^0_{npq}(\alpha)$, $V^0_{npq}(\alpha)$ and $Q^0_{npq}(\alpha)$ as

$$P^{1i}_{npq}(\alpha)=-(n^2-1)P^0_{npq}(\alpha)+\frac{n(2n-1)(n+2)}{(2n+1)}V^0_{npq}(\alpha)$$
$$+\sum_{s=1}^{q}\begin{bmatrix}n+s\\n+m\end{bmatrix}\Bigg[\frac{(n-1)[ns(n+s-2ns-2)-m^2(2ns-4s-4n+2)]}{s(n+s)(2s-1)}P^0_{s(q-s)(p-n+1)}(3-\alpha)$$
$$+\frac{n^2(2n-1)(n+2)}{(2n+3)(n+1)}P^0_{s(q-s)(p-n-1)}(3-\alpha)$$
$$+\frac{n(2n-1)(n-1)}{(2s+1)}V^0_{s(q-s-2)(p-n+1)}(3-\alpha)$$
$$-\frac{4m(n-1)(2n-1)}{3}Q^0_{s(q-s-1)(p-n+1)}(3-\alpha)\Bigg], \quad (67)$$

$$V^{1i}_{npq}(\alpha)=\frac{(2n+1)}{(2n-1)}P^{1i}_{npq}(\alpha), \quad (68)$$

$$Q^{1i}_{npq}(\alpha)=(n+2)Q^0_{npq}(\alpha)+\sum_{s=1}^{q}\begin{bmatrix}n+s\\n+m\end{bmatrix}\Bigg[\frac{s(n-1)}{n+1}Q^0_{s(q-s-1)(p-n)}(3-\alpha)$$
$$-\frac{3m(n-1)}{2ns(n+1)}P^0_{s(q-s-1)(p-n)}(3-\alpha)\Bigg], \quad (69)$$

where the superscript i denotes that these quantities are referring to initial values.

In order to determine the ten scalar functions in the resistance matrix, it is sufficient to consider four different cases, where sphere 1 is assumed to move translationally or rotationally and sphere 2 is at rest.

Case (i): In order to evaluate X^A_{11} and X^A_{21}, sphere 1 is assumed to move in the positive z-direction with a uniform velocity U and sphere 2 is at rest.

Applying the results in (29)–(31) in conjunction with the expansions (58)–(60), we have

$$m = 0, \tag{70}$$

$$P^0_{n00}(1) = \delta_{1n}, \tag{71}$$

$$V^0_{n00}(1) = \delta_{1n}, \tag{72}$$

$$Q^0_{n00}(1) = 0, \tag{73}$$

which will initiate the iterative calculations in (61) and (62). After obtaining the values of $P^0_{npq}(\alpha)$ and $V^0_{npq}(\alpha)$ as the results of the iterative calculations, they are used to provide the initial values of the first-order quantities $P^1_{npq}(\alpha)$ and $V^1_{npq}(\alpha)$ through (67)–(69). Then, again, the iterative calculations in (61) and (62) are carried out for the first-order quantities.

The expansion (58) gives the values of $p^0_{01}(\alpha)$ and $p^1_{01}(\alpha)$, which determine the forces experienced by sphere 1 and sphere 2 through (24). According to the definition of the resistance matrix the functions X^A_{11} and X^A_{21} are found to be

$$X^A_{11} = -6\pi a_1 \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} [P^0_{1pq}(1) + K\kappa_0 P^1_{1pq}(1)]\, \xi_1^p \xi_2^q, \tag{74}$$

and

$$X^A_{21} = -6\pi a_2 \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} [P^0_{1pq}(2) + K\kappa_0 P^1_{1pq}(2)]\, \xi_2^p \xi_1^q. \tag{75}$$

For the purpose of applications, only a certain finite number of terms needs to be retained in (74) and (75). In this presentation, we limit the degree of approximation to $p+q \leqslant 11$, which should be sufficiently accurate up to the point of overlapping Knudsen layers on the two spheres. To this extent the results for X^A_{11} and X^A_{21} are

$$\begin{aligned} X^A_{11} = -6\pi a_1 \{ & [1 + \tfrac{9}{4}\xi_1 \xi_2 - \tfrac{3}{2}\xi_1^3 \xi_2 + \tfrac{81}{16}\xi_1^2 \xi_2^2 + \tfrac{9}{4}\xi_1 \xi_2^3 + \tfrac{1}{4}\xi_1^5 \xi_2 + \tfrac{27}{16}\xi_1^4 \xi_2^2 \\ & + \tfrac{281}{64}\xi_1^3 \xi_2^3 + \tfrac{81}{8}\xi_1^2 \xi_2^4 + \tfrac{9}{4}\xi_1 \xi_2^5 + \tfrac{9}{4}\xi_1^6 \xi_2^2 + \tfrac{303}{16}\xi_1^5 \xi_2^3 + \tfrac{5409}{256}\xi_1^4 \xi_2^4 \\ & + \tfrac{1131}{64}\xi_1^3 \xi_2^5 + \tfrac{243}{16}\xi_1^2 \xi_2^6 + \tfrac{9}{4}\xi_1 \xi_2^7 + \tfrac{9}{4}\xi_1^8 \xi_2^2 + \tfrac{81}{4}\xi_1^7 \xi_2^3 + \tfrac{10701}{256}\xi_1^6 \xi_2^4 \\ & + \tfrac{115849}{1024}\xi_1^5 \xi_2^5 + \tfrac{4761}{64}\xi_1^4 \xi_2^6 + \tfrac{1227}{32}\xi_1^3 \xi_2^7 + \tfrac{81}{4}\xi_1^2 \xi_2^8 + \tfrac{9}{4}\xi_1 \xi_2^9] + K\kappa_0[1 + \tfrac{9}{4}\xi_1^2 \\ & + \tfrac{9}{2}\xi_1 \xi_2 - \tfrac{3}{2}\xi_1^4 + \tfrac{33}{8}\xi_1^3 \xi_2 + \tfrac{351}{16}\xi_1^2 \xi_2^2 + \tfrac{9}{2}\xi_1 \xi_2^3 + \tfrac{1}{4}\xi_1^6 + \tfrac{39}{8}\xi_1^5 \xi_2 \\ & + \tfrac{1383}{64}\xi_1^4 \xi_2^2 + \tfrac{929}{16}\xi_1^3 \xi_2^3 + \tfrac{333}{8}\xi_1^2 \xi_2^4 + \tfrac{9}{2}\xi_1 \xi_2^5 + \tfrac{9}{2}\xi_1^7 \xi_2 + \tfrac{1161}{16}\xi_1^6 \xi_2^2 \\ & + \tfrac{12681}{64}\xi_1^5 \xi_2^3 + \tfrac{49665}{256}\xi_1^4 \xi_2^4 + \tfrac{2589}{16}\xi_1^3 \xi_2^5 + \tfrac{981}{16}\xi_1^2 \xi_2^6 + \tfrac{9}{2}\xi_1 \xi_2^7 + \tfrac{9}{2}\xi_1^9 \xi_2 \\ & + 81\xi_1^8 \xi_2^2 + \tfrac{21069}{64}\xi_1^7 \xi_2^3 + \tfrac{878873}{1024}\xi_1^6 \xi_2^4 + \tfrac{576075}{512}\xi_1^5 \xi_2^5 + \tfrac{40983}{64}\xi_1^4 \xi_2^6 \\ & + \tfrac{2523}{8}\xi_1^3 \xi_2^7 + 81\xi_1^2 \xi_2^8 + \tfrac{9}{2}\xi_1 \xi_2^9]\}, \end{aligned} \tag{76}$$

and

$$\begin{aligned} X^A_{21} = -6\pi a_2 \{ & [-\tfrac{3}{2}\xi_1 + \tfrac{1}{2}\xi_2^2 \xi_1 - \tfrac{27}{8}\xi_2 \xi_1^2 + \tfrac{1}{2}\xi_1^3 - \tfrac{9}{4}\xi_2^3 \xi_1^2 - \tfrac{243}{32}\xi_2^2 \xi_1^3 \\ & - \tfrac{9}{4}\xi_2 \xi_1^4 - \tfrac{9}{4}\xi_2^5 \xi_1^2 - \tfrac{405}{32}\xi_2^4 \xi_1^3 - \tfrac{1515}{128}\xi_2^3 \xi_1^4 - \tfrac{405}{32}\xi_2^2 \xi_1^5 - \tfrac{9}{4}\xi_2 \xi_1^6 - \tfrac{9}{4}\xi_2^7 \xi_1^2 \\ & - \tfrac{567}{32}\xi_2^6 \xi_1^3 - \tfrac{461}{16}\xi_2^5 \xi_1^4 - \tfrac{26163}{512}\xi_2^4 \xi_1^5 - \tfrac{461}{16}\xi_2^3 \xi_1^6 - \tfrac{567}{32}\xi_2^2 \xi_1^7 - \tfrac{9}{4}\xi_2 \xi_1^8 \\ & - \tfrac{9}{4}\xi_2^9 \xi_1^2 - \tfrac{729}{32}\xi_2^8 \xi_1^3 - \tfrac{6807}{128}\xi_2^7 \xi_1^4 - \tfrac{67275}{512}\xi_2^6 \xi_1^5 - \tfrac{319899}{2048}\xi_2^5 \xi_1^6 - \tfrac{67275}{512}\xi_2^4 \xi_1^7 \\ & - \tfrac{6807}{128}\xi_2^3 \xi_1^8 - \tfrac{729}{32}\xi_2^2 \xi_1^9 - \tfrac{9}{4}\xi_2 \xi_1^{10}] + K\kappa_0[-\tfrac{3}{2}\xi_1 - \tfrac{3}{2}\xi_2^{-1}\xi_1^2 + \tfrac{1}{2}\xi_2^2 \xi_1 \\ & - \tfrac{21}{4}\xi_2 \xi_1^2 - \tfrac{21}{4}\xi_1^3 + \tfrac{1}{2}\xi_2^{-1}\xi_1^4 - \tfrac{9}{2}\xi_2^3 \xi_1^2 - \tfrac{1017}{32}\xi_2^2 \xi_1^3 - \tfrac{1017}{32}\xi_2 \xi_1^4 - \tfrac{9}{2}\xi_2^5 \end{aligned}$$

$$-\tfrac{9}{2}\xi_2^5\,\xi_1^2-\tfrac{1647}{32}\xi_2^4\,\xi_1^3-\tfrac{885}{8}\xi_2^3\,\xi_1^4-\tfrac{885}{8}\xi_2^2\,\xi_1^5-\tfrac{1647}{32}\xi_2\,\xi_1^6-\tfrac{9}{2}\xi_1^7-\tfrac{9}{2}\xi_2^7\,\xi_1^2$$
$$-\tfrac{2277}{32}\xi_2^6\,\xi_1^3-\tfrac{7657}{32}\xi_2^5\,\xi_1^4-\tfrac{219327}{512}\xi_2^4\,\xi_1^5-\tfrac{219327}{512}\xi_2^3\,\xi_1^6-\tfrac{7657}{32}\xi_2^2\,\xi_1^7$$
$$-\tfrac{2277}{32}\xi_2\,\xi_1^8-\tfrac{9}{2}\xi_1^9-\tfrac{9}{2}\xi_2^9\,\xi_1^2-\tfrac{2907}{32}\xi_2^8\,\xi_1^3-\tfrac{1671}{4}\xi_2^7\,\xi_1^4-\tfrac{554199}{512}\xi_2^6\,\xi_1^5$$
$$-\tfrac{1901547}{1024}\xi_2^5\,\xi_1^6-\tfrac{1901547}{1024}\xi_2^4\,\xi_1^7-\tfrac{554199}{512}\xi_2^3\,\xi_1^8-\tfrac{1671}{4}\xi_2^2\,\xi_1^9-\tfrac{2907}{32}\xi_2\,\xi_1^{10}$$
$$-\tfrac{9}{2}\xi_1^{11}]\}. \tag{77}$$

The appearance of a negative exponent in (77) is not erroneous. Note that after being multiplied by the two factors $-6\pi a_2$ and $K\kappa_0$, (77) contains no negative exponents.

Case (ii): In order to evaluate the functions Y_{11}^A, Y_{21}^A, Y_{11}^B and Y_{21}^B, sphere 1 is assumed to move in the positive x-direction with a uniform velocity U and sphere 2 is at rest.

In this case,

$$m = 1, \tag{78}$$

$$P_{n00}^0(1) = \delta_{1n}, \tag{79}$$

$$V_{n00}^0(1) = \delta_{1n}, \tag{80}$$

$$Q_{n00}^0(1) = 0. \tag{81}$$

The non-axisymmetrical nature of this case requires that the iterative calculations in (61)–(63) be done simultaneously. Besides the forces experienced by the two spheres as obtained in the same manner as in case (i), the resulting values of $q_{01}^0(\alpha)$ and $q_{01}^1(\alpha)$ will determine the torques experienced by the two spheres through (25). Thus the functions Y_{11}^A, Y_{21}^A, Y_{11}^B and Y_{21}^B are found to be

$$Y_{11}^A = -6\pi a_1 \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} [P_{1pq}^0(1)+K\kappa_0 P_{1pq}^1(1)]\,\xi_1^p\,\xi_2^q, \tag{82}$$

$$Y_{21}^A = 6\pi a_2 \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} [P_{1pq}^0(2)+K\kappa_0 P_{1pq}^1(2)]\,\xi_2^p\,\xi_1^q, \tag{83}$$

$$Y_{11}^B = -8\pi a_1^2 \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} [Q_{1pq}^0(1)+K\kappa_0 Q_{1pq}^1(1)]\,\xi_1^p\,\xi_2^q, \tag{84}$$

and

$$Y_{21}^B = -8\pi a_2^2 \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} [Q_{1pq}^0(2)+K\kappa_0 Q_{1pq}^1(2)]\,\xi_2^p\,\xi_1^q. \tag{85}$$

The results of our calculations give

$$Y_{11}^A = -6\pi a_1\{[1+\tfrac{9}{16}\xi_1\,\xi_2+\tfrac{3}{8}\xi_1^3\,\xi_2+\tfrac{81}{256}\xi_1^2\,\xi_2^2+\tfrac{9}{8}\xi_1\,\xi_2^3+\tfrac{1}{16}\xi_1^5\,\xi_2+\tfrac{27}{32}\xi_1^4\,\xi_2^2$$
$$+\tfrac{1241}{4096}\xi_1^3\,\xi_2^3+\tfrac{81}{64}\xi_1^2\,\xi_2^4+\tfrac{9}{8}\xi_1\,\xi_2^5+\tfrac{279}{256}\xi_1^6\,\xi_2^2+\tfrac{4261}{2048}\xi_1^5\,\xi_2^3+\tfrac{126369}{65536}\xi_1^4\,\xi_2^4$$
$$-\tfrac{117}{2048}\xi_1^3\,\xi_2^5+\tfrac{81}{32}\xi_1^2\,\xi_2^6+\tfrac{9}{8}\xi_1\,\xi_2^7+\tfrac{9}{8}\xi_1^8\,\xi_2^2+\tfrac{7857}{4096}\xi_1^7\,\xi_2^3+\tfrac{98487}{16384}\xi_1^6\,\xi_2^4$$
$$+\tfrac{10548393}{1048576}\xi_1^5\,\xi_2^5+\tfrac{67617}{8192}\xi_1^4\,\xi_2^6-\tfrac{351}{2048}\xi_1^3\,\xi_2^7+\tfrac{243}{64}\xi_1^2\,\xi_2^8+\tfrac{9}{8}\xi_1\,\xi_2^9]+K\kappa_0[1$$
$$+\tfrac{9}{16}\xi_1^2+\tfrac{9}{8}\xi_1\,\xi_2+\tfrac{3}{8}\xi_1^4+\tfrac{273}{128}\xi_1^3\,\xi_2+\tfrac{1107}{256}\xi_1^2\,\xi_2^2+\tfrac{9}{4}\xi_1\,\xi_2^3+\tfrac{1}{16}\xi_1^6$$
$$+\tfrac{33}{16}\xi_1^5\,\xi_2+\tfrac{21003}{4096}\xi_1^4\,\xi_2^2+\tfrac{6425}{1024}\xi_1^3\,\xi_2^3+\tfrac{603}{64}\xi_1^2\,\xi_2^4+\tfrac{9}{4}\xi_1\,\xi_2^5+\tfrac{279}{128}\xi_1^7\,\xi_2$$
$$+\tfrac{28407}{2048}\xi_1^6\,\xi_2^2+\tfrac{330897}{16384}\xi_1^5\,\xi_2^3+\tfrac{613125}{65536}\xi_1^4\,\xi_2^4-\tfrac{7659}{512}\xi_1^3\,\xi_2^5+\tfrac{495}{32}\xi_1^2\,\xi_2^6+\tfrac{9}{4}\xi_1\,\xi_2^7$$
$$+\tfrac{9}{4}\xi_1^9\,\xi_2+\tfrac{65043}{4096}\xi_1^8\,\xi_2^2+\tfrac{161343}{4096}\xi_1^7\,\xi_2^3+\tfrac{96864141}{1048576}\xi_1^6\,\xi_2^4+\tfrac{57610107}{524288}\xi_1^5\,\xi_2^5$$
$$+\tfrac{328257}{8192}\xi_1^4\,\xi_2^6-\tfrac{15201}{512}\xi_1^3\,\xi_2^7+\tfrac{1377}{64}\xi_1^2\,\xi_2^8+\tfrac{81}{8}\xi_1\,\xi_2^9]\}, \tag{86}$$

$$\begin{aligned} Y_{21}^{A} = 6\pi a_2\{[&\tfrac{3}{4}\xi_1+\tfrac{1}{4}\xi_2^2\,\xi_1+\tfrac{27}{64}\xi_2\,\xi_1^2+\tfrac{1}{4}\xi_1^3+\tfrac{63}{64}\xi_2^3\,\xi_1^2+\tfrac{243}{1024}\xi_2^2\,\xi_1^3 \\ &+\tfrac{63}{64}\xi_2\,\xi_1^4+\tfrac{9}{8}\xi_2^5\,\xi_1^2+\tfrac{1053}{1024}\xi_2^4\,\xi_1^3+\tfrac{19083}{16384}\xi_2^3\,\xi_1^4+\tfrac{1053}{1024}\xi_2^2\,\xi_1^5+\tfrac{9}{8}\xi_2\,\xi_1^6+\tfrac{9}{8}\xi_2^7\,\xi_1^2 \\ &+\tfrac{567}{256}\xi_2^6\,\xi_1^3+\tfrac{60443}{16384}\xi_2^5\,\xi_1^4+\tfrac{766179}{262144}\xi_2^4\,\xi_1^5+\tfrac{60443}{16384}\xi_2^3\,\xi_1^6+\tfrac{567}{256}\xi_2^2\,\xi_1^7+\tfrac{9}{8}\xi_2\,\xi_1^8 \\ &+\tfrac{9}{8}\xi_2^9\,\xi_1^2+\tfrac{891}{256}\xi_2^8\,\xi_1^3+\tfrac{22071}{4096}\xi_2^7\,\xi_1^4+\tfrac{2744505}{262144}\xi_2^6\,\xi_1^5+\tfrac{95203835}{4194304}\xi_2^5\,\xi_1^6 \\ &+\tfrac{2744505}{262144}\xi_2^4\,\xi_1^7+\tfrac{22071}{4096}\xi_2^3\,\xi_1^8+\tfrac{891}{256}\xi_2^2\,\xi_1^9+\tfrac{9}{8}\xi_2\,\xi_1^{10}]+K\kappa_0[\tfrac{3}{4}\xi_1+\tfrac{3}{4}\xi_2^{-1}\xi_1^2 \\ &+\tfrac{1}{4}\xi_2^2\,\xi_1+\tfrac{51}{32}\xi_2\,\xi_1^2+\tfrac{51}{32}\xi_1^3+\tfrac{1}{4}\xi_2^{-1}\xi_1^4+\tfrac{63}{32}\xi_2^3\,\xi_1^2+\tfrac{4761}{1024}\xi_2^2\,\xi_1^3+\tfrac{4761}{1024}\xi_2\,\xi_1^4 \\ &+\tfrac{63}{32}\xi_1^5+\tfrac{9}{4}\xi_2^5\,\xi_1^2+\tfrac{10071}{1024}\xi_2^4\,\xi_1^3+\tfrac{40143}{4096}\xi_2^3\,\xi_1^4+\tfrac{40143}{4096}\xi_2^2\,\xi_1^5+\tfrac{10071}{1024}\xi_2\,\xi_1^6 \\ &+\tfrac{9}{4}\xi_1^7+\tfrac{9}{4}\xi_2^7\,\xi_1^2+\tfrac{4005}{256}\xi_2^6\,\xi_1^3+\tfrac{123947}{4096}\xi_2^5\,\xi_1^4+\tfrac{9633423}{262144}\xi_2^4\,\xi_1^5+\tfrac{9633423}{262144}\xi_2^3\,\xi_1^6 \\ &+\tfrac{123947}{4096}\xi_2^2\,\xi_1^7+\tfrac{4005}{256}\xi_2\,\xi_1^8+\tfrac{9}{4}\xi_1^9+\tfrac{9}{4}\xi_2^9\,\xi_1^2+\tfrac{5553}{256}\xi_2^8\,\xi_1^3+\tfrac{54147}{1024}\xi_2^7\,\xi_1^4 \\ &+\tfrac{25022877}{262144}\xi_2^6\,\xi_1^5+\tfrac{439303785}{2097152}\xi_2^5\,\xi_1^6+\tfrac{439303785}{2097152}\xi_2^4\,\xi_1^7+\tfrac{25022877}{262144}\xi_2^3\,\xi_1^8 \\ &+\tfrac{54147}{1024}\xi_2^2\,\xi_1^9+\tfrac{5553}{256}\xi_2\,\xi_1^{10}+\tfrac{9}{4}\xi_1^{11}]\}, \end{aligned} \tag{87}$$

$$\begin{aligned} Y_{11}^{B} = -8\pi a_1^2\{[&-\tfrac{9}{16}\xi_1^2\,\xi_2-\tfrac{3}{16}\xi_1^4\,\xi_2-\tfrac{81}{256}\xi_1^3\,\xi_2^2-\tfrac{9}{16}\xi_1^2\,\xi_2^3-\tfrac{189}{256}\xi_1^5\,\xi_2^2 \\ &-\tfrac{8409}{4096}\xi_1^4\,\xi_2^3-\tfrac{243}{256}\xi_1^3\,\xi_2^4-\tfrac{9}{16}\xi_1^2\,\xi_2^5-\tfrac{27}{32}\xi_1^7\,\xi_2^2-\tfrac{3159}{4096}\xi_1^6\,\xi_2^3-\tfrac{283041}{65536}\xi_1^5\,\xi_2^4 \\ &-\tfrac{30525}{4096}\xi_1^4\,\xi_2^5-\tfrac{405}{256}\xi_1^3\,\xi_2^6-\tfrac{9}{16}\xi_1^2\,\xi_2^7-\tfrac{27}{32}\xi_1^9\,\xi_2^2-\tfrac{1701}{1024}\xi_1^8\,\xi_2^3-\tfrac{614481}{65536}\xi_1^7\,\xi_2^4 \\ &-\tfrac{4579497}{1048576}\xi_1^6\,\xi_2^5-\tfrac{536679}{65536}\xi_1^5\,\xi_2^6-\tfrac{73989}{4096}\xi_1^4\,\xi_2^7-\tfrac{567}{256}\xi_1^3\,\xi_2^8-\tfrac{9}{16}\xi_1^2\,\xi_2^9] \\ &+K\kappa_0[-\tfrac{9}{4}\xi_1^2\,\xi_2-\tfrac{9}{16}\xi_1\,\xi_2^2-\tfrac{9}{8}\xi_1^4\,\xi_2-\tfrac{453}{256}\xi_1^3\,\xi_2^2-\tfrac{369}{128}\xi_1^2\,\xi_2^3-\tfrac{27}{16}\xi_1\,\xi_2^4 \\ &-\tfrac{1323}{256}\xi_1^5\,\xi_2^2-\tfrac{28251}{2048}\xi_1^4\,\xi_2^3-\tfrac{44667}{4096}\xi_1^3\,\xi_2^4-\tfrac{387}{64}\xi_1^2\,\xi_2^5-\tfrac{45}{16}\xi_1\,\xi_2^6-\tfrac{243}{32}\xi_1^7\,\xi_2^2 \\ &-\tfrac{4023}{512}\xi_1^6\,\xi_2^3-\tfrac{2132919}{65536}\xi_1^5\,\xi_2^4-\tfrac{1015641}{16384}\xi_1^4\,\xi_2^5-\tfrac{185025}{4096}\xi_1^3\,\xi_2^6-\tfrac{1503}{128}\xi_1^2\,\xi_2^7 \\ &-\tfrac{63}{16}\xi_1\,\xi_2^8-\tfrac{297}{32}\xi_1^9\,\xi_2^2-\tfrac{9369}{512}\xi_1^8\,\xi_2^3-\tfrac{5856921}{65536}\xi_1^7\,\xi_2^4-\tfrac{9495345}{131072}\xi_1^6\,\xi_2^5 \\ &-\tfrac{83005533}{1048576}\xi_1^5\,\xi_2^6-\tfrac{5161509}{32768}\xi_1^4\,\xi_2^7-\tfrac{563283}{4096}\xi_1^3\,\xi_2^8-\tfrac{639}{32}\xi_1^2\,\xi_2^9-\tfrac{81}{16}\xi_1\,\xi_2^{10}]\}, \end{aligned} \tag{88}$$

and

$$\begin{aligned} Y_{21}^{B} = -8\pi a_2^2\{[&-\tfrac{3}{4}\xi_2\,\xi_1-\tfrac{27}{64}\xi_2^2\,\xi_1^2-\tfrac{27}{32}\xi_2^4\,\xi_1^2-\tfrac{243}{1024}\xi_2^3\,\xi_1^3-\tfrac{9}{16}\xi_2^2\,\xi_1^4 \\ &-\tfrac{27}{32}\xi_2^6\,\xi_1^2-\tfrac{243}{256}\xi_2^5\,\xi_1^3-\tfrac{77451}{16384}\xi_2^4\,\xi_1^4-\tfrac{405}{512}\xi_2^3\,\xi_1^5-\tfrac{9}{16}\xi_2^2\,\xi_1^6-\tfrac{27}{32}\xi_2^8\,\xi_1^2 \\ &-\tfrac{243}{128}\xi_2^7\,\xi_1^3-\tfrac{59553}{8192}\xi_2^6\,\xi_1^4-\tfrac{1125603}{262144}\xi_2^5\,\xi_1^5-\tfrac{11001}{1024}\xi_2^4\,\xi_1^6-\tfrac{729}{512}\xi_2^3\,\xi_1^7 \\ &-\tfrac{9}{16}\xi_2^2\,\xi_1^8]+K\kappa_0[-\tfrac{9}{4}\xi_2^2-\tfrac{3}{4}\xi_2\,\xi_1-\tfrac{27}{16}\xi_2^3\,\xi_1-\tfrac{37}{32}\xi_2^2\,\xi_1^2-\tfrac{81}{16}\xi_2^5\,\xi_1 \\ &-\tfrac{2943}{1024}\xi_2^4\,\xi_1^2-\tfrac{3033}{1024}\xi_2^3\,\xi_1^3-\tfrac{9}{4}\xi_2^2\,\xi_1^4-\tfrac{27}{4}\xi_2^7\,\xi_1-\tfrac{2133}{256}\xi_2^6\,\xi_1^2-\tfrac{255681}{8192}\xi_2^5\,\xi_1^3 \\ &-\tfrac{93651}{4096}\xi_2^4\,\xi_1^4-\tfrac{3177}{512}\xi_2^3\,\xi_1^5-\tfrac{27}{8}\xi_2^2\,\xi_1^6-\tfrac{135}{16}\xi_2^9\,\xi_1-\tfrac{2403}{128}\xi_2^8\,\xi_1^2-\tfrac{65385}{1024}\xi_2^7\,\xi_1^3 \\ &-\tfrac{15502005}{262144}\xi_2^6\,\xi_2^4-\tfrac{22525551}{262144}\xi_2^5\,\xi_1^5-\tfrac{4581}{64}\xi_2^4\,\xi_1^6-\tfrac{6255}{512}\xi_2^3\,\xi_1^7-\tfrac{9}{2}\xi_2^2\,\xi_1^8]\}. \end{aligned} \tag{89}$$

Case (iii): In order to evaluate the functions X_{11}^{C} and X_{21}^{C}, sphere 1 is assumed to rotate with a fixed angular velocity Ω in the positive z-direction and sphere 2 is at rest.

Setting $U = a_1 \Omega$ and applying the results in (37)–(39), we have

$$m = 0, \tag{90}$$

$$P_{n00}^{0}(1) = 0, \tag{91}$$

$$V^0_{n00}(1) = 0, \tag{92}$$

$$Q^0_{n00}(1) = -\mathrm{i}\delta_{1n}. \tag{93}$$

Only the recurrence relation (63) is needed in the calculation, and the functions X^C_{11} and X^C_{21} are given as

$$X^C_{11} = -\mathrm{i}8\pi a_1^3 \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} [Q^0_{1pq}(1) + K\kappa_0 Q^1_{1pq}(1)]\, \xi_1^p \xi_2^q, \tag{94}$$

and

$$X^C_{21} = \mathrm{i}8\pi a_1 a_2^2 \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} [Q^0_{1pq}(2) + K\kappa_0 Q^1_{1pq}(2)]\, \xi_2^p \xi_1^q. \tag{95}$$

Our numerical calculations result in

$$X^C_{11} = -8\pi a_1^3\{[1 + \xi_1^3\xi_2^3 + 3\xi_1^3\xi_2^5 + 6\xi_1^3\xi_2^7] + K\kappa_0[3 + 3\xi_1^4\xi_2^2 + 6\xi_1^3\xi_2^3 + 15\xi_1^4\xi_2^4 + 18\xi_1^3\xi_2^5 + 42\xi_1^4\xi_2^6 + 36\xi_1^3\xi_2^7]\}, \tag{96}$$

and

$$X^C_{21} = 8\pi a_1 a_2^2\{[\xi_2\xi_1^2 + \xi_2^4\xi_1^5 + 3\xi_2^6\xi_1^5 + 3\xi_2^4\xi_1^7] + K\kappa_0[3\xi_2\xi_1^2 + 3\xi_1^3 + 6\xi_2^4\xi_1^5 + 6\xi_2^3\xi_1^6 + 18\xi_2^6\xi_1^5 + 24\xi_2^5\xi_1^6 + 42\xi_2^4\xi_1^7 + 18\xi_2^3\xi_1^8]\}. \tag{97}$$

Case (iv): In order to evaluate Y^C_{11} and Y^C_{21}, sphere 1 is assumed to rotate with a fixed angular velocity Ω in the positive x-direction and sphere 2 is at rest.

Again setting $U = a_1\Omega$, we have

$$m = 1, \tag{98}$$

$$P^0_{n00}(1) = 0, \tag{99}$$

$$V^0_{n00}(1) = 0, \tag{100}$$

$$Q^0_{n00}(1) = -\mathrm{i}\delta_{1n}. \tag{101}$$

All three relations (61)–(63) are needed for carrying out the iterative calculations, and the functions Y^C_{11} and Y^C_{21} are given by

$$Y^C_{11} = -\mathrm{i}8\pi a_1^3 \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} [Q^0_{1pq}(1) + K\kappa_0 Q^1_{1pq}(1)]\, \xi_1^p \xi_2^q, \tag{102}$$

and

$$Y^C_{21} = -\mathrm{i}8\pi a_1 a_2^2 \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} [Q^0_{1pq}(2) + K\kappa_0 Q^1_{1pq}(2)]\, \xi_2^p \xi_1^q. \tag{103}$$

Explicitly, from the results of our numerical calculation these functions are found to be

$$\begin{aligned} Y^C_{11} = -8\pi a_1^3\{&[1 + \tfrac{3}{4}\xi_1^3\xi_2 + \tfrac{27}{64}\xi_1^4\xi_2^2 + 4\xi_1^3\xi_2^3 + \tfrac{27}{32}\xi_1^6\xi_2^2 + \tfrac{243}{1024}\xi_1^5\xi_2^3 \\ &+ \tfrac{27}{32}\xi_1^4\xi_2^4 + \tfrac{39}{4}\xi_1^3\xi_2^5 + \tfrac{27}{32}\xi_1^8\xi_2^2 + \tfrac{243}{256}\xi_1^7\xi_2^3 + \tfrac{151179}{16384}\xi_1^6\xi_2^4 + \tfrac{243}{256}\xi_1^5\xi_2^5 \\ &+ \tfrac{81}{64}\xi_1^4\xi_2^6 + 18\xi_1^3\xi_2^7] + K\kappa_0[3 + \tfrac{3}{4}\xi_1^4 + \tfrac{9}{2}\xi_1^3\xi_2 + \tfrac{27}{32}\xi_1^5\xi_2 + \tfrac{957}{64}\xi_1^4\xi_2^2 \\ &+ 24\xi_1^3\xi_2^3 + \tfrac{27}{16}\xi_1^7\xi_2 + \tfrac{8505}{1024}\xi_1^6\xi_2^2 + \tfrac{675}{128}\xi_1^5\xi_2^3 + \tfrac{1749}{32}\xi_1^4\xi_2^4 + \tfrac{117}{2}\xi_1^3\xi_2^5 \\ &+ \tfrac{27}{16}\xi_1^9\xi_2 + \tfrac{3105}{256}\xi_1^8\xi_2^2 + \tfrac{190059}{4096}\xi_1^7\xi_2^3 + \tfrac{1438371}{16384}\xi_1^6\xi_2^4 + \tfrac{243}{16}\xi_1^5\xi_2^5 \\ &+ \tfrac{8631}{64}\xi_1^4\xi_2^6 + 108\xi_1^3\xi_2^7]\}, \end{aligned} \tag{104}$$

and

$$
\begin{aligned}
Y^C_{21} = -8\pi a_1 a_2^2\{&[\tfrac{1}{2}\xi_2\,\xi_1^2+\tfrac{9}{16}\xi_2^2\,\xi_1^3+\tfrac{9}{16}\xi_2^4\,\xi_1^3+\tfrac{81}{256}\xi_2^3\,\xi_1^4+\tfrac{9}{16}\xi_2^2\,\xi_1^5+\tfrac{9}{16}\xi_2^6\,\xi_1^3 \\
&+\tfrac{243}{256}\xi_2^5\,\xi_1^4+\tfrac{6439}{4096}\xi_2^4\,\xi_1^5+\tfrac{243}{256}\xi_2^3\,\xi_1^6+\tfrac{9}{16}\xi_2^2\,\xi_1^7+\tfrac{9}{16}\xi_2^8\,\xi_1^3+\tfrac{405}{256}\xi_2^7\,\xi_1^4 \\
&-\tfrac{10947}{4096}\xi_2^6\,\xi_1+\tfrac{518049}{65536}\xi_2^5\,\xi_1^6-\tfrac{10947}{4096}\xi_2^4\,\xi_1^7+\tfrac{405}{256}\xi_2^3\,\xi_1^8+\tfrac{9}{16}\xi_2^2\,\xi_1^9] \\
&+K\kappa_0[\tfrac{3}{2}\xi_2\,\xi_1^2+\tfrac{3}{2}\xi_1^3+\tfrac{9}{4}\xi_2^2\,\xi_1^3+\tfrac{9}{4}\xi_2\,\xi_1^4+\tfrac{9}{4}\xi_2^4\,\xi_1^3+\tfrac{1269}{256}\xi_2^3\,\xi_1^4+\tfrac{1269}{256}\xi_2^2\,\xi_1^5 \\
&+\tfrac{9}{4}\xi_2\,\xi_1^6+\tfrac{9}{4}\xi_2^6\,\xi_1^3+\tfrac{2367}{256}\xi_2^5\,\xi_1^4+\tfrac{5709}{2048}\xi_2^4\,\xi_1^5+\tfrac{5709}{2048}\xi_2^3\,\xi_1^6+\tfrac{2367}{256}\xi_2^2\,\xi_1^7 \\
&+\tfrac{9}{4}\xi_2\,\xi_1^8+\tfrac{9}{4}\xi_2^8\,\xi_1^3+\tfrac{3465}{256}\xi_2^7\,\xi_1^4-\tfrac{3681}{2048}\xi_2^6\,\xi_1^5+\tfrac{2225127}{65536}\xi_2^5\,\xi_1^6-\tfrac{2225127}{65536}\xi_2^4\,\xi_1^7 \\
&-\tfrac{3681}{2048}\xi_2^3\,\xi_1^8+\tfrac{3465}{256}\xi_2^2\,\xi_1^9+\tfrac{9}{4}\xi_2\,\xi_1^{10}]\}.
\end{aligned} \tag{105}
$$

Note that in this case the forces experienced by sphere 1 and sphere 2 can also be obtained through the resulting coefficients $p^0_{1pq}(\alpha)$ and $p^1_{1pq}(\alpha)$ just as in case (i) and case (ii). It, therefore, provides an alternative way to evaluate the functions Y^B_{11} and Y^B_{21}. The numerical results show that the values of the functions Y^B_{11} and Y^B_{21} thus obtained agree with the ones given in case (ii), as predicted by the symmetrical properties of the resistance matrix in the general case with slip boundary conditions.

We also note the symmetrical properties in the results of X^A_{21}, Y^A_{21}, X^C_{21} and Y^C_{21}. Owing to the two-sphere geometry, the expressions for the functions X^A_{12}, Y^A_{12}, X^C_{12} and Y^C_{12} can be readily obtained by interchanging the sphere indices 1 and 2 in these expressions. The expressions for the functions X^A_{12}, Y^A_{12}, X^C_{12} and Y^C_{12} thus obtained are shown to be identical to the expressions for X^A_{21}, Y^A_{21}, X^C_{21} and Y^C_{21}, respectively. This fact once again is in agreement with the prediction from the symmetrical properties of the resistance matrix.

It is shown that in all four cases studied, in the limit of $K = 0$, our results recover the solutions given by Jeffrey & Onishi as physically expected.

4.2. *The mobility matrix*

If the specified quantities are the forces and torques exerted by the fluid on the spheres, inverting (47) gives

$$
\begin{bmatrix} \boldsymbol{U}_1-\boldsymbol{U}_0 \\ \boldsymbol{U}_2-\boldsymbol{U}_0 \\ \boldsymbol{\Omega}_1-\boldsymbol{\Omega}_0 \\ \boldsymbol{\Omega}_2-\boldsymbol{\Omega}_0 \end{bmatrix} = \mu^{-1} \begin{bmatrix} a_{11} & a_{12} & \tilde{b}_{11} & \tilde{b}_{12} \\ a_{21} & a_{22} & \tilde{b}_{21} & \tilde{b}_{22} \\ b_{11} & b_{12} & c_{11} & c_{12} \\ b_{21} & b_{22} & c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \end{bmatrix}. \tag{106}
$$

The square matrix is the mobility matrix, which is the inverse of the resistance matrix. All the symmetry conditions obeyed by the resistance matrix also hold for the mobility matrix. The tensors in the mobility matrix, therefore, can be written as

$$a_{\alpha\beta} = x^a_{\alpha\beta}\hat{\boldsymbol{k}}\hat{\boldsymbol{k}} + y^a_{\alpha\beta}(\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}\hat{\boldsymbol{\jmath}}), \tag{107}$$

$$b_{\alpha\beta} = y^b_{\alpha\beta}(\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}-\hat{\boldsymbol{\jmath}}\hat{\boldsymbol{\imath}}), \tag{108}$$

$$c_{\alpha\beta} = x^c_{\alpha\beta}\hat{\boldsymbol{k}}\hat{\boldsymbol{k}} + y^c_{\alpha\beta}(\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}\hat{\boldsymbol{\jmath}}). \tag{109}$$

Again, only ten scalar functions have to be evaluated to completely determine the mobility matrix, which can be conveniently chosen as: x^a_{11}, x^a_{21}, y^a_{11}, y^a_{21}, y^b_{11}, y^b_{21}, x^c_{11}, x^c_{21}, y^c_{11} and y^c_{21}.

The problem raised here is that from the given forces and torques acting on the spheres, the velocities and angular velocities of the spheres are to be obtained from the solutions of the velocity field.

The mobility functions can be obtained either by an inversion of the resistance matrix or by a direct derivation. As indicated by Yoon & Kim (1987), however, there are some disadvantages of the inversion procedure mainly caused by the singular behaviour of the resistance functions at small separations which makes the inversion problem ill-conditioned. We have also noticed that the amount of numerical work involved in the inversion of the resistance matrix is considerably greater than that required in a direct derivation of the mobility functions. Therefore, here we adopt a direct method similar to the one used by Jeffrey & Onishi (1984) to approach the mobility problem. In one particular case, where the inversion procedure is relatively easy, we also perform the inversion of some of the resistance functions to verify, in part, the direct method.

In the resistance matrix problem, the given velocities and angular velocities of the two spheres are related only to the zero-order Hilbert solution of the velocity field; the first-order Hilbert solution being subsequently determined from the zero-order solution. In the mobility matrix problem, on the other hand, the given forces and torques are coupled to both the zero-order and first-order Hilbert solutions of the velocity field. This feature requires a somewhat different approach to the calculation of the mobility matrix from the one used to calculate the resistance matrix.

For the purpose of calculating the mobility matrix, some special cases will be chosen, where a force (or a torque) in either the z- or x-direction is acting on sphere 1 and no force or torque is acting on sphere 2. In such cases, from the symmetrical properties of the mobility matrix, the velocities and angular velocities of the spheres will be in the direction of one of the axes in Cartesian coordinates. Furthermore, they are found to be related to the three scalar functions in (55)–(57) by

$$X^0_{mn}(\alpha) = U(\alpha)\,\delta_{n1}, \tag{110}$$

$$\Psi^0_{mn}(\alpha) = 0, \tag{111}$$

$$Z^0_{mn}(\alpha) = 2a_\alpha\,\Omega(\alpha)\,\delta_{n1}, \tag{112}$$

where $U(\alpha)$ and $\Omega(\alpha)$ are the magnitudes of the velocity and angular velocity of sphere α, respectively, and $m = 0$ or 1 depending on the direction of the velocity or angular velocity.

Substitution of (110)–(112) into the right-hand sides of (55)–(57) will eliminate the three scalar functions from (55)–(57) and lead to general relations between the zero-order Hilbert solution of the velocity field and the velocities and angular velocities of the spheres.

Next, in order to determine $U(\alpha)$ and $\Omega(\alpha)$, the following double power series expansions are needed:

$$U(\alpha) = (-1)^{3-\alpha}(-1)^{(1+m)(\alpha-1)}U\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}U_{pq}(\alpha)\,\xi^p_\alpha\,\xi^q_{3-\alpha}, \tag{113}$$

$$\Omega(\alpha) = (-1)^{(1+m)(\alpha-1)}\mathrm{i}\frac{U}{a_\alpha}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Omega_{pq}(\alpha)\,\xi^p_\alpha\,\xi^q_{3-\alpha}, \tag{114}$$

where U is a quantity of the dimension of velocity defined in terms of the given force or torque acting on sphere 1 as follows.

If the given quantity is a force with magnitude F acting on sphere 1, U is defined as

$$U = -\frac{F}{6\pi\mu a_1}; \tag{115}$$

if the given quantity is a torque with magnitude T acting on sphere 1, U is defined as

$$U = -\frac{T}{8\pi\mu a_1^2}. \tag{116}$$

Substituting (113)–(114) into (55)–(57) with the right-hand sides replaced by $U(\alpha)$ and $\Omega(\alpha)$ from (110) and (112), then equating the coefficients of the same powers of ξ_α and $\xi_{3-\alpha}$, we obtain the following relations.

For $n = 1$,

$$U_{pq}(\alpha) = P^0_{1pq}(\alpha) - \sum_{s=1}^{q} \begin{bmatrix} s+1 \\ m+1 \end{bmatrix} \bigg[\frac{3(2m^2-s)}{4s(2s-1)} P^0_{s(q-s)\,p}(3-\alpha) + \tfrac{1}{4} P^0_{s(q-s)\,(p-2)}(3-\alpha) + \frac{3}{4(2s+1)} V^0_{s(q-s-2)\,p}(3-\alpha) - mQ^0_{s(q-s-1)\,p}(3-\alpha) \bigg], \tag{117}$$

$$\Omega_{pq}(\alpha) = Q^0_{1pq}(\alpha) - \sum_{s=1}^{q} \begin{bmatrix} s+1 \\ m+1 \end{bmatrix} \bigg[\tfrac{1}{2} s Q^0_{s(q-s-1)\,(p-1)}(3-\alpha) - \frac{3m}{4s} P^0_{s(q-s)\,(p-1)}(3-\alpha) \bigg]. \tag{118}$$

Note that the recurrence relations (61) and (63) do not apply to the case of $n = 1$ with $\delta = 0$; instead we have relations (117) and (118).

The initial values needed for carrying out the recursive computations in (61)–(63) are provided by the prescribed values of the force and torque, as explained below.

There are two basic cases to be considered depending on whether the given quantity is a force or a torque. In case I, the prescribed quantity is a force F in the z- or x-direction acting on sphere 1 and there are no other forces or torques acting on either one of the spheres. According to (24) and (25) we have

$$F = -4\pi\mu a_1 [p^0_{m1}(1) + Kp^1_{m1}(1)], \tag{119}$$

$$0 = -4\pi\mu a_2 [p^0_{m1}(2) + Kp^1_{m1}(2)], \tag{120}$$

$$0 = -8\pi\mu a_1^2 [q^0_{m1}(1) + Kq^1_{m1}(1)], \tag{121}$$

$$0 = -8\pi\mu a_2^2 [q^0_{m1}(2) + Kq^1_{m1}(2)]. \tag{122}$$

Substituting the double series expansions of $p^\delta_{m1}(\alpha)$ and $q^\delta_{m1}(\alpha)$ in (58) and (60) and the quantity U defined in (115) into (119)–(122), then equating the coefficients of the same powers of ξ_α and $\xi_{3-\alpha}$, we have the following relations referring to the initial values that are to be used for the recursive computations in (61)–(63):

$$P^0_{1pq}(1) + \kappa_0 K P^1_{1pq}(1) = \delta_{p0}\,\delta_{q0}, \tag{123}$$

$$P^0_{1pq}(2) + \kappa_0 K P^1_{1pq}(2) = 0, \tag{124}$$

$$Q^0_{1pq}(\alpha) + \kappa_0 K Q^1_{1pq}(\alpha) = 0 \quad (\alpha = 1, 2). \tag{125}$$

In case II, the prescribed quantity is a torque T in the z- or x-direction acting on sphere 1 and there are no other forces or torques acting on either one of the spheres, which leads to

$$0 = -4\pi\mu a_1 [p^0_{m1}(1) + Kp^1_{m1}(1)], \tag{126}$$

$$0 = -4\pi\mu a_2 [p^0_{m1}(2) + Kp^1_{m1}(2)], \tag{127}$$

$$T = -8\pi\mu a_1^2 [q^0_{m1}(1) + Kq^1_{m1}(1)], \tag{128}$$

$$0 = -8\pi\mu a_2^2 [q^0_{m1}(2) + Kq^1_{m1}(2)]. \tag{129}$$

The same substitutions as before lead to the following relations:

$$P^0_{1pq}(\alpha)+\kappa_0 K P^1_{1pq}(\alpha) = 0 \quad (\alpha = 1,2), \tag{130}$$

$$Q^0_{1pq}(1)+\kappa_0 K Q^1_{1pq}(1) = -\mathrm{i}\delta_{p0}\,\delta_{q0}, \tag{131}$$

$$Q^0_{1pq}(2)+\kappa_0 K Q^1_{1pq}(2) = 0. \tag{132}$$

From (117) and (118) it is clear that the determination of $U_{pq}(\alpha)$ and $\Omega_{pq}(\alpha)$, and therefore the mobility functions, relies on the determination of the coefficients $P^0_{npq}(\alpha)$, $V^0_{npq}(\alpha)$ and $Q^0_{npq}(\alpha)$.

However, unlike the resistance matrix problem, the known conditions for the coefficients $P^0_{npq}(\alpha)$, $V^0_{npq}(\alpha)$ and $Q^0_{npq}(\alpha)$ are coupled to those for the coefficients $P^1_{npq}(\alpha)$, $V^1_{npq}(\alpha)$ and $Q^1_{npq}(\alpha)$ through (123)–(125) or (130)–(132). Although this coupled problem can be rather involved, as given below, we developed a simplified procedure that starts from a decomposition of the coefficients $P^0_{npq}(\alpha)$, $V^0_{npq}(\alpha)$, $Q^0_{npq}(\alpha)$, $P^1_{npq}(\alpha)$, $V^1_{npq}(\alpha)$ and $Q^1_{npq}(\alpha)$ to different orders of the Knudsen number, Kn, in a power series expansions of K, allowing us to decouple the problem to a great extent.

In the small-Knudsen-number analysis, the mobility functions, like the other physical quantities, being expanded in power series of K, consist of terms of order 1 (K^0) and order K. Therefore, the coefficients $P^0_{npq}(\alpha)$, $V^0_{npq}(\alpha)$ and $Q^0_{npq}(\alpha)$ inevitably consist of terms of order 1 and order K, as well. Actually, this physical consideration manifests itself through (123)–(125) in case I or (130)–(132) in case II. Equations (123)–(125) show that, for example, the initial values of $P^0_{npq}(\alpha)$ and $Q^0_{npq}(\alpha)$, and therefore the coefficients $P^0_{npq}(\alpha)$, $V^0_{npq}(\alpha)$ and $Q^0_{npq}(\alpha)$ themselves, must contain parts of order K as well as of order 1 otherwise these conditions cannot be met.

As a result, we write these coefficients as

$$P^{\delta}_{npq}(\alpha) = P^{\delta(0)}_{npq}(\alpha)+P^{\delta(1)}_{npq}(\alpha), \tag{133}$$

$$V^{\delta}_{npq}(\alpha) = V^{\delta(0)}_{npq}(\alpha)+V^{\delta(1)}_{npq}(\alpha), \tag{134}$$

$$Q^{\delta}_{npq}(\alpha) = Q^{\delta(0)}_{npq}(\alpha)+Q^{\delta(1)}_{npq}(\alpha), \tag{135}$$

where one more superscript (0) or (1) is added to distinguish the parts of order 1 and order K.

The numerical calculations start from the calculation of $P^{0(0)}_{npq}(\alpha)$, $V^{0(0)}_{npq}(\alpha)$ and $Q^{0(0)}_{npq}(\alpha)$, whose initial values are derived from (26)–(29) in case I or from (130)–(132) in case II as

case I:

$$P^{0(0)}_{1pq}(1) = \delta_{p0}\,\delta_{q0}, \tag{136}$$

$$P^{0(0)}_{1pq}(2) = 0, \tag{137}$$

$$Q^{0(0)}_{1pq}(\alpha) = 0 \quad (\alpha = 1,2). \tag{138}$$

case II:

$$P^{0(0)}_{1pq}(\alpha) = 0 \quad (\alpha = 1,2), \tag{139}$$

$$Q^{0(0)}_{1pq}(1) = -\mathrm{i}\delta_{p0}\,\delta_{q0}, \tag{140}$$

$$Q^{0(0)}_{1pq}(2) = 0. \tag{141}$$

Using these initial values in the recurrence relations (61)–(63) and carrying out the recursive computations, the entire set of coefficients $P^{0(0)}_{npq}(\alpha)$, $V^{0(0)}_{npq}(\alpha)$ and $Q^{0(0)}_{npq}(\alpha)$ are obtained. Note that (61) and (63) are only used under the condition $n > 1$.

The results for $P^{0(0)}_{npq}(\alpha)$, $V^{0(0)}_{npq}(\alpha)$ and $Q^{0(0)}_{npq}(\alpha)$, which are quantities related to the zero-order velocity field, are used to provide the initial values of the quantities $P^{1(0)}_{npq}(\alpha)$, $V^{1(0)}_{npq}(\alpha)$ and $Q^{1(0)}_{npq}(\alpha)$, which are quantities related to the first-order velocity field, through (67)–(69). Note that these two groups of coefficients all belong to zero-order quantities in the K-expansions, and therefore should be related in such a way. Then the recursive computations in (61) and (63) are carried out to obtain the entire set of coefficients $P^{1(0)}_{npq}(\alpha)$, $V^{1(0)}_{npq}(\alpha)$ and $Q^{1(0)}_{npq}(\alpha)$. Actually, the procedure described above is the same as in the calculations of the resistance matrix from the coefficients $P^{0}_{npq}(\alpha)$, $V^{0}_{npq}(\alpha)$ and $Q^{0}_{npq}(\alpha)$ to the coefficients $P^{1}_{npq}(\alpha)$, $V^{1}_{npq}(\alpha)$ and $Q^{1}_{npq}(\alpha)$.

The initial values of $P^{0(1)}_{1pq}(\alpha)$, $V^{0(1)}_{1pq}(\alpha)$ and $Q^{0(1)}_{1pq}(\alpha)$ are related to the resulting values of $P^{1(0)}_{1pq}(\alpha)$, $V^{1(0)}_{1pq}(\alpha)$ and $Q^{1(0)}_{1pq}(\alpha)$ through (123)–(125) in case I or (130)–(132) in case II. Both cases lead to the same relations:

$$P^{0(1)}_{1pq}(\alpha) = -\kappa_0 K P^{1(0)}_{1pq}(\alpha) \quad (\alpha = 1, 2), \tag{142}$$

and

$$Q^{0(1)}_{1pq}(\alpha) = -\kappa_0 K Q^{1(0)}_{1pq}(\alpha) \quad (\alpha = 1, 2). \tag{143}$$

Again, these initial values are used to carry out the recursive computations in (61)–(63) (with (61) and (63) only used when $n > 1$) to obtain the coefficients $P^{0(1)}_{npq}(\alpha)$, $V^{0(1)}_{npq}(\alpha)$ and $Q^{0(1)}_{npq}(\alpha)$. Then (133)–(135) give the values of $P^{0}_{npq}(\alpha)$, $V^{0}_{npq}(\alpha)$ and $Q^{0}_{npq}(\alpha)$, which are used to calculate the quantities $U_{pq}(\alpha)$ and $\Omega_{pq}(\alpha)$ through (117) and (118).

In order to determine the ten scalar functions in the mobility matrix, it is sufficient to consider four different cases, which are to be introduced and treated in the following.

Case (i): In order to evaluate the functions x^a_{11} and x^a_{21} a force F in the positive z-direction is assumed to act on sphere 1 and no force or torque is acting on sphere 2.

This is an axisymmetrical case with $m = 0$. Equations (136)–(138) provide the initial values to start the iterative calculations. After carrying out all the iterative calculations described above, we obtain the values of the coefficients $P^{0(0)}_{npq}(\alpha)$, $V^{0(0)}_{npq}(\alpha)$, $P^{0(1)}_{npq}(\alpha)$ and $V^{0(1)}_{npq}(\alpha)$ (all the coefficients Q_{npq} vanish). Then the velocities of sphere 1 and sphere 2 are determined through (113) and (117) with the coefficients $P^{0}_{npq}(\alpha)$ and $V^{0}_{npq}(\alpha)$ expressed by (133) and (134).

The definition of the mobility functions leads to the following expressions for x^a_{11} and x^a_{21}:

$$x^a_{11} = -\frac{1}{6\pi a_1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} U_{pq}(1)\, \xi_1^p \xi_2^q, \tag{144}$$

and

$$x^a_{21} = -\frac{1}{6\pi a_1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} U_{pq}(2)\, \xi_2^p \xi_1^q. \tag{145}$$

As discussed in the resistance matrix case, only a certain finite number of terms needs to be retained in (144) and (145) and we limit the degree of approximation to $p+q \leqslant 11$. To this extent, the numerical results of our computations give the functions x^a_{11} and x^a_{21} as

$$\begin{aligned} x^a_{11} = -\frac{1}{6\pi a_1}\{[&1 - \tfrac{15}{4}\xi_1 \xi_2^3 + \tfrac{15}{2}\xi_1^3 \xi_2^3 - 2\xi_1 \xi_2^5 - \tfrac{15}{4}\xi_1^5 \xi_2^3 + \tfrac{33}{2}\xi_1^3 \xi_2^5 - \tfrac{9}{4}\xi_1 \xi_2^7 \\ &- \tfrac{35}{2}\xi_1^5 \xi_2^5 - \tfrac{187}{2}\xi_1^4 \xi_2^6 + 30\xi_1^3 \xi_2^7 - \tfrac{9}{4}\xi_1 \xi_2^9] - K\kappa_0[1 + \tfrac{45}{4}\xi_1^2 \xi_2^2 - \tfrac{45}{2}\xi_1^4 \xi_2^2 \\ &- 15\xi_1^3 \xi_2^3 + 10\xi_1^2 \xi_2^4 + \tfrac{45}{4}\xi_1^6 \xi_2^2 + 15\xi_1^5 \xi_2^3 - \tfrac{165}{2}\xi_1^4 \xi_2^4 - 33\xi_1^3 \xi_2^5 + \tfrac{63}{4}\xi_1^2 \xi_2^6 \\ &+ \tfrac{175}{2}\xi_1^6 \xi_2^4 + \tfrac{1265}{2}\xi_1^5 \xi_2^5 + \tfrac{285}{4}\xi_1^4 \xi_2^6 - 60\xi_1^3 \xi_2^7 + \tfrac{81}{4}\xi_1^2 \xi_2^8]\}, \end{aligned} \tag{146}$$

and

$$x^a_{21} = -\frac{1}{6\pi a_1}\{[\tfrac{3}{2}\xi_1 - \tfrac{1}{2}\xi_2^2\,\xi_1 - \tfrac{1}{2}\xi_1^3 + \tfrac{75}{4}\xi_2^3\,\xi_1^4 - \tfrac{15}{4}\xi_2^5\,\xi_1^4$$
$$- \tfrac{15}{4}\xi_2^3\,\xi_1^6 + \tfrac{15}{2}\xi_2^7\,\xi_1^4 - \tfrac{453}{4}\xi_2^5\,\xi_1^6 + \tfrac{15}{2}\xi_2^3\,\xi_1^8] - K\kappa_0[\xi_2\,\xi_1^2 + \xi_1^3 - \tfrac{225}{4}\xi_2^3\,\xi_1^4$$
$$- \tfrac{225}{4}\xi_2^2\,\xi_1^5 + \tfrac{45}{4}\xi_2^5\,\xi_1^4 + \tfrac{75}{4}\xi_2^4\,\xi_1^5 + \tfrac{75}{4}\xi_2^3\,\xi_1^6 + \tfrac{45}{4}\xi_2^2\,\xi_1^7 - \tfrac{45}{2}\xi_2^7\,\xi_1^4$$
$$- \tfrac{105}{2}\xi_2^6\,\xi_1^5 + \tfrac{2265}{4}\xi_2^5\,\xi_1^6 + \tfrac{2265}{4}\xi_2^4\,\xi_1^7 - \tfrac{105}{2}\xi_2^3\,\xi_1^8 - \tfrac{45}{2}\xi_2^2\,\xi_1^9]\}. \tag{147}$$

Case (ii): In order to evaluate the functions y^a_{11}, y^a_{21}, y^b_{11} and y^b_{21}, a force F in the positive x-direction is assumed to act on sphere 1 and no force or torque is acting on sphere 2.

In this non-axisymmetrical case, $m = 1$ and (136)–(138) provide the initial values for the iterative calculations. These calculations give the values of $P^{0(0)}_{npq}(\alpha)$, $V^{0(0)}_{npq}(\alpha)$, $Q^{0(0)}_{npq}(\alpha)$, $P^{0(1)}_{npq}(\alpha)$, $V^{0(1)}_{npq}(\alpha)$ and $Q^{0(1)}_{npq}(\alpha)$, which are used to determine the velocities and angular velocities of the two spheres through (113)–(114), (117) and (118).

Then the functions y^a_{11}, y^a_{21}, y^b_{11} and y^b_{21} can be expressed as

$$y^a_{11} = -\frac{1}{6\pi a_1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} U_{pq}(1)\,\xi_1^p\,\xi_2^q, \tag{148}$$

$$y^a_{21} = \frac{1}{6\pi a_1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} U_{pq}(2)\,\xi_2^p\,\xi_1^q, \tag{149}$$

$$y^b_{11} = -\frac{1}{6\pi a_1^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \Omega_{pq}(1)\,\xi_1^p\,\xi_2^q, \tag{150}$$

and

$$y^b_{21} = -\frac{1}{6\pi a_1\,a_2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \Omega_{pq}(2)\,\xi_2^p\,\xi_1^q. \tag{151}$$

Our numerical results give

$$y^a_{11} = -\frac{1}{6\pi a_1}\{[1 - \tfrac{17}{16}\xi_1\,\xi_2^5 - \tfrac{5}{4}\xi_1^5\,\xi_2^3 + \tfrac{9}{8}\xi_1^3\,\xi_2^5 - \tfrac{9}{8}\xi_1\,\xi_2^7 - \tfrac{105}{16}\xi_1^5\,\xi_2^5$$
$$+ \tfrac{27}{8}\xi_1^3\,\xi_2^7 - \tfrac{9}{8}\xi_1\,\xi_2^9] - K\kappa_0[1 + \tfrac{85}{16}\xi_1^2\,\xi_2^4 + \tfrac{15}{4}\xi_1^6\,\xi_2^2 + 5\xi_1^5\,\xi_2^3 - \tfrac{45}{8}\xi_1^4\,\xi_2^4$$
$$- \tfrac{9}{4}\xi_1^3\,\xi_2^5 + \tfrac{63}{8}\xi_1^2\,\xi_2^6 + \tfrac{525}{16}\xi_1^6\,\xi_2^4 + \tfrac{105}{4}\xi_1^5\,\xi_2^5 - \tfrac{189}{8}\xi_1^4\,\xi_2^6 - \tfrac{27}{4}\xi_1^3\,\xi_2^7$$
$$+ \tfrac{81}{8}\xi_1^2\,\xi_2^8]\}, \tag{152}$$

$$y^a_{21} = \frac{1}{6\pi a_1}\{[-\tfrac{3}{4}\xi_1 - \tfrac{1}{4}\xi_2^2\,\xi_1 - \tfrac{1}{4}\xi_1^3 - \tfrac{35}{8}\xi_2^7\,\xi_1^4 - \tfrac{553}{128}\xi_2^5\,\xi_1^6 - \tfrac{35}{8}\xi_2^3\,\xi_1^8]$$
$$- K\kappa_0[\tfrac{1}{2}\xi_2\,\xi_1^2 + \tfrac{1}{2}\xi_1^3 + \tfrac{105}{8}\xi_2^7\,\xi_1^4 + \tfrac{245}{8}\xi_2^6\,\xi_1^5 - \tfrac{2765}{128}\xi_2^5\,\xi_1^6 - \tfrac{2765}{128}\xi_2^4\,\xi_1^7$$
$$+ \tfrac{245}{8}\xi_2^3\,\xi_1^8 + \tfrac{105}{8}\xi_2^2\,\xi_1^9]\}, \tag{153}$$

$$y^b_{11} = -\frac{1}{6\pi a_1^2}\{[\tfrac{15}{8}\xi_1^4\,\xi_2^3 + \tfrac{9}{16}\xi_1^2\,\xi_2^5 + \tfrac{105}{16}\xi_1^4\,\xi_2^5 + \tfrac{9}{16}\xi_1^2\,\xi_2^7 + \tfrac{63}{4}\xi_1^4\,\xi_2^7$$
$$- \tfrac{9}{16}\xi_1^2\,\xi_2^9] - K\kappa_0[-\tfrac{45}{8}\xi_1^5\,\xi_2^2 - \tfrac{15}{4}\xi_1^4\,\xi_2^3 - \tfrac{45}{16}\xi_1^3\,\xi_2^4 - \tfrac{525}{16}\xi_1^5\,\xi_2^4 - \tfrac{105}{8}\xi_1^4\,\xi_2^5$$
$$- \tfrac{63}{16}\xi_1^3\,\xi_2^6 - \tfrac{441}{4}\xi_1^5\,\xi_2^6 - \tfrac{63}{2}\xi_1^4\,\xi_2^7 - \tfrac{81}{16}\xi_1^3\,\xi_2^8]\}, \tag{154}$$

and

$$y^b_{21} = -\frac{1}{6\pi a_1 a_2}\{[\tfrac{3}{4}\xi_2\,\xi_1 + \tfrac{105}{16}\xi_2^6\,\xi_1^4 - \tfrac{75}{16}\xi_2^4\,\xi_1^6] - K\kappa_0[-\tfrac{315}{16}\xi_2^6\,\xi_1^4 - \tfrac{525}{16}\xi_2^5\,\xi_1^5 + \tfrac{375}{16}\xi_2^4\,\xi_1^6 + \tfrac{225}{16}\xi_2^3\,\xi_1^7]\}. \tag{155}$$

Case (iii): In order to evaluate the functions x^c_{11} and x^c_{21}, a torque T in the positive z-direction is assumed to act on sphere 1 and no force or torque is acting on sphere 2.

In this axisymmetrical case, $m = 1$ and all the coefficients P_{npq} and V_{npq} vanish. The initial values for iteration are given by (140) and (141). The functions x^c_{11} and x^c_{21} can be expressed as

$$x^c_{11} = -\frac{\mathrm{i}}{8\pi a_1^3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Omega_{pq}(1)\,\xi_1^p\,\xi_2^q, \tag{156}$$

and

$$x^c_{21} = \frac{\mathrm{i}}{8\pi a_1^2 a_2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Omega_{pq}(2)\,\xi_2^p\,\xi_1^q. \tag{157}$$

The numerical results are

$$x^c_{11} = -\frac{1}{8\pi a_1^3}\{[1 - 3\xi_1^3\,\xi_2^5 - 6\xi_1^3\,\xi_2^7] - K\kappa_0[3 + 15\xi_1^4\,\xi_2^4 + 42\xi_1^4\,\xi_2^6]\}, \tag{158}$$

and

$$x^c_{21} = -\frac{1}{8\pi a_1^2 a_2}\xi_2\,\xi_1^2. \tag{159}$$

In this particular case, as mentioned at the beginning, we can also readily carry out an inversion process from the results of the resistance functions to derive these mobility functions by invoking the following relation (Kim & Mifflin 1985):

$$\begin{bmatrix} x^c_{11} & x^c_{12} \\ x^c_{21} & x^c_{22} \end{bmatrix} = \begin{bmatrix} X^C_{11} & X^C_{12} \\ X^C_{21} & X^C_{22} \end{bmatrix}^{-1}.$$

The results for the mobility functions x^c_{11} and x^c_{21} derived from these two different methods turn out to be in agreement within the limit of $p+q \leqslant 11$ in the double power series expansions.

Case (iv): In order to evaluate the functions y^c_{11} and y^c_{21} a torque T in the positive x-direction is assumed to act on sphere 1 and no force or torque is acting on sphere 2.

In this case $m = 1$ and (139)–(141) give the initial values for iteration. The functions y^c_{11} and y^c_{21} can be expressed as

$$y^c_{11} = -\frac{\mathrm{i}}{8\pi a_1^3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Omega_{pq}(1)\,\xi_1^p\,\xi_2^q, \tag{160}$$

and

$$y^c_{21} = -\frac{\mathrm{i}}{8\pi a_1^2 a_2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Omega_{pq}(2)\,\xi_2^p\,\xi_1^q. \tag{161}$$

The numerical results are

$$y^c_{11} = -\frac{1}{8\pi a_1^3}\{[1 - \tfrac{15}{4}\xi_1^3\,\xi_2^3 - \tfrac{39}{4}\xi_1^3\,\xi_2^5 - 18\xi_1^3\,\xi_2^7] - K\kappa_0[3 + \tfrac{45}{4}\xi_1^4\,\xi_2^2 + \tfrac{195}{4}\xi_1^4\,\xi_2^4 + 126\xi_1^4\,\xi_2^6]\}, \tag{162}$$

and

$$y^c_{21} = -\frac{1}{6\pi a_1^2 a_2}\{[-\tfrac{1}{2}\xi_2\,\xi_1^2 + \tfrac{75}{8}\xi_2^4\,\xi_1^5 + 15\xi_2^6\,\xi_1^5 + 15\xi_2^4\,\xi_1^7] - K\kappa_0[-\tfrac{225}{8}\xi_2^4\,\xi_1^5 - \tfrac{225}{8}\xi_2^3\,\xi_1^6 - 45\xi_2^6\,\xi_1^5 - 75\xi_2^5\,\xi_1^6 - 75\xi_2^4\,\xi_1^7 - 45\xi_2^3\,\xi_1^8]\}. \tag{163}$$

$Kn = 0$ l/a	$A(l)$ (exact)	$A_3(l)$	$A_7(l)$	$A_{11}(l)$	$B(l)$ (exact)	$B_3(l)$	$B_7(l)$	$B_{11}(l)$
2.0	1.5500	1.6250	1.6231	1.5201	1.3799	1.4375	1.4209	1.4182
2.1	1.5363	1.6063	1.5817	1.5221	1.3918	1.4111	1.3987	1.3966
2.6749	1.4662	1.5085	1.4694	1.4660	1.3029	1.3065	1.3036	1.3032
3.0	1.4320	1.4630	1.4328	1.4320	1.2668	1.2685	1.2671	1.2669
4.0	1.3472	1.3594	1.3472	1.3472	1.1950	1.1953	1.1951	1.1950
6.0	1.2427	1.2454	1.2427	1.2427	1.1273	1.1273	1.1273	1.1273
8.0	1.1847	1.1855	1.1847	1.1847	1.0947	1.0947	1.0947	1.0947

$Kn = 0.01$ l/a	$A_3(l)$	$A_7(l)$	$A_{11}(l)$	$B_3(l)$	$B_7(l)$	$B_{11}(l)$
2.0	1.6363	1.6289	1.5353	1.4476	1.4295	1.4273
2.1	1.6173	1.5894	1.5352	1.4211	1.4073	1.4055
2.6749	1.5185	1.4796	1.4765	1.3160	1.3123	1.3119
3.0	1.4726	1.4429	1.4423	1.2779	1.2758	1.2756
4.0	1.3687	1.3568	1.3568	1.2045	1.2039	1.2039
6.0	1.2545	1.2518	1.2518	1.1364	1.1363	1.1363
8.0	1.1946	1.1937	1.1937	1.1038	1.1037	1.1037

$Kn = 0.1$ l/a	$A_3(l)$	$A_7(l)$	$A_{11}(l)$	$B_3(l)$	$B_7(l)$	$B_{11}(l)$
2.0	1.7376	1.6811	1.6722	1.5388	1.5072	1.5090
2.1	1.7158	1.6582	1.6537	1.5109	1.4847	1.4855
2.6749	1.6080	1.5716	1.5717	1.4013	1.3903	1.3902
3.0	1.5597	1.5340	1.5342	1.3619	1.3544	1.3544
4.0	1.4522	1.4428	1.4429	1.2868	1.2838	1.2838
6.0	1.3363	1.3342	1.3342	1.2178	1.2169	1.2169
8.0	1.2760	1.2753	1.2753	1.1850	1.1846	1.1846

$Kn = 0.5$ l/a	$A_3(l)$	$A_7(l)$	$A_{11}(l)$	$B_3(l)$	$B_7(l)$	$B_{11}(l)$
2.0	2.1879	1.9133	2.2803	1.9441	1.8523	1.8723
2.1	2.1538	1.9641	2.1801	1.9100	1.8283	1.8413
2.6749	2.0059	1.9802	1.9948	1.7803	1.7369	1.7385
3.0	1.9466	1.9388	1.9426	1.7355	1.7040	1.7045
4.0	1.8237	1.8252	1.8253	1.6526	1.6389	1.6389
6.0	1.6998	1.7006	1.5755	1.5797	1.5755	1.5755
8.0	1.6376	1.6379	1.6379	1.5459	1.5441	1.5441

TABLE 1. Numerical values of the functions $A(l)$ and $B(l)$ defined in (164) and (165). The values of l/a given are taken directly from tables 6 and 8A of Goldman *et al.* (1966), table 2 of Cooley & O'Neill (1969) and table 2 of O'Neill & Majumdar (1970).

We note that the results for the mobility functions show the same symmetric properties as the resistance matrix. Also it is shown that in all the four cases above, in the limit of $K = 0$, our results recover the solutions given by Jeffrey & Onishi as physically expected.

A comparison of our numerical results with the exact results given by Stimson & Jeffery (1926), Goldman, Cox & Brenner (1966), Cooley & O'Neill (1969) and O'Neill & Majumdar (1970) in the case of two equal spheres with radius a is shown in table 1, where the coefficients $A(l)$ and $B(l)$ are defined as

$$A(l) = -\frac{1}{6\pi a}(x_{11}^a + x_{12}^a); \tag{164}$$

$$B(l) = -\frac{1}{6\pi a}(y_{11}^a + y_{12}^a). \qquad (165)$$

The subscripts 3, 7 and 11 in table 1 denote that these coefficients are obtained from our power series expansions with terms up to orders l^{-3}, l^{-7} and l^{-11}, respectively, and the $A(l)$ (exact) and $B(l)$ (exact) are the exact results. In the case of $Kn = 0$, our numerical results should be the same as those obtainable from Jeffrey & Onishi (1984) and from Felderhof (1977) (only to the order of l^{-7} in Felderhof's results). To show the effect of different values of the Knudsen number, we list the numerical values of the coefficients $A(l)$ and $B(l)$ for $Kn = 0$, 0.01, 0.1, 0.5, respectively.

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